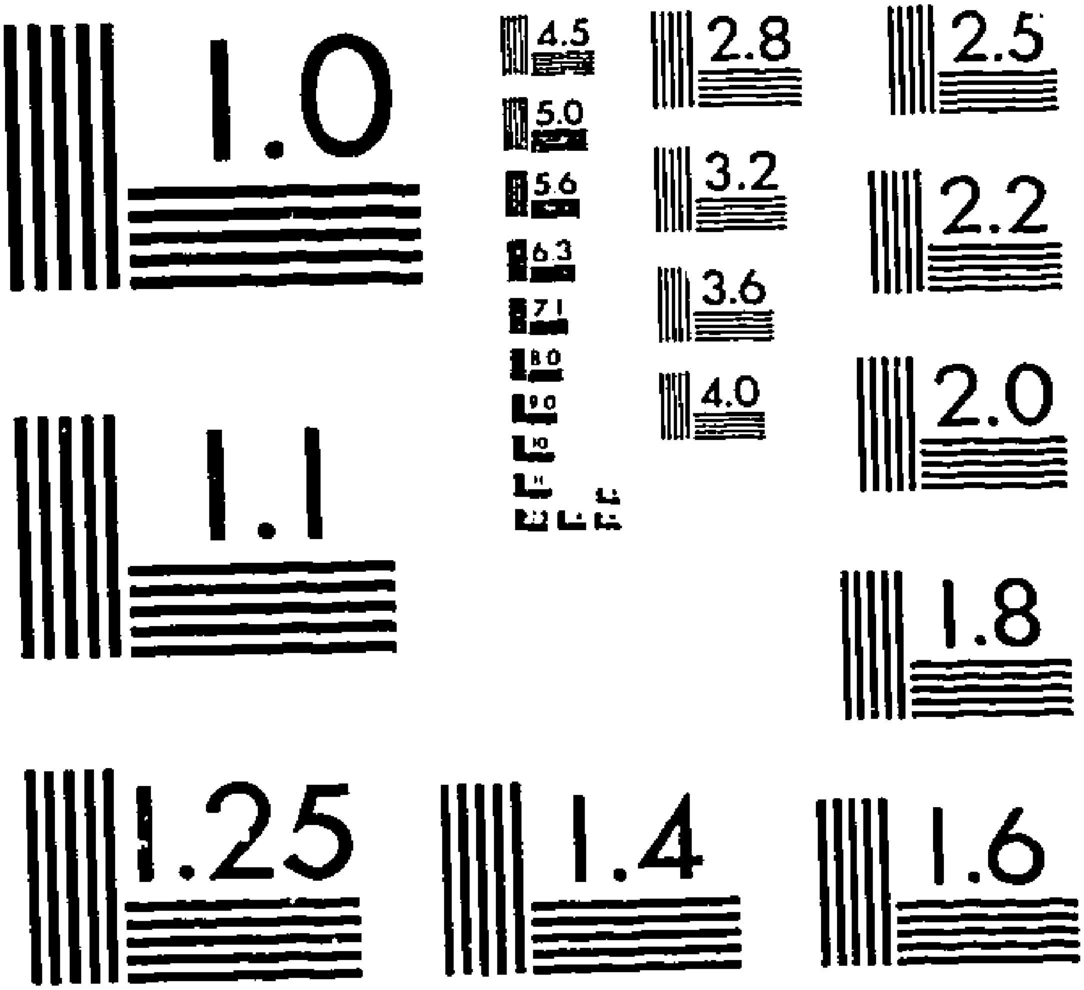

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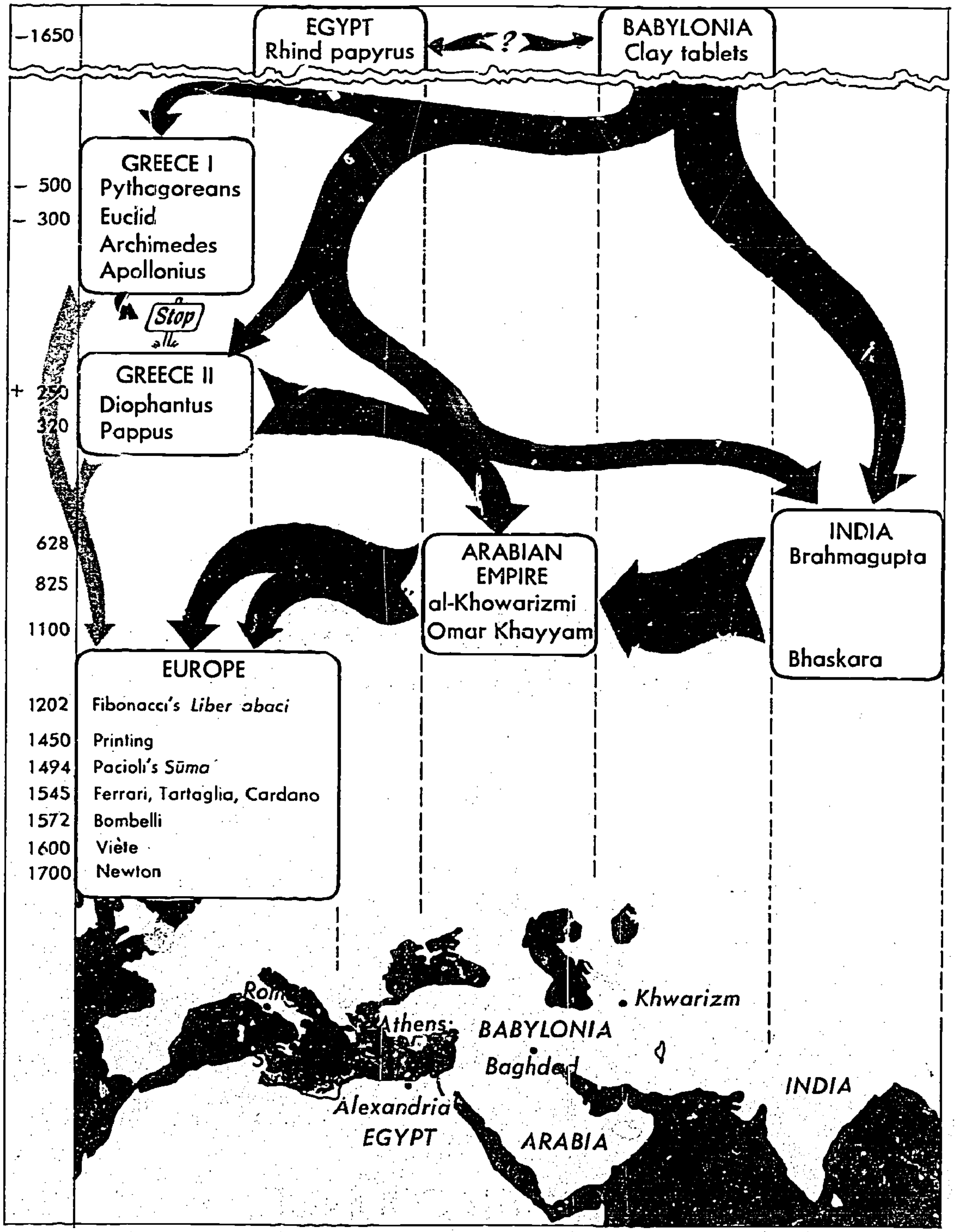
ABSTRACT

This is a reprint of the historical capsules dealing with algebra from the 31st Yearbook of NCTM,"Historical Topics for the Mathematics Classroom." Included are such themes as the change from a geometric to an algebraic solution of problems, the development of algebraic symbolism, the algebraic contributions of different countries, the origin and development of topics in algebra, and the search for generality and abstract structures. (Author/JG)

Mainstreams in the Flow of Algebra



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Preface

This booklet is a separate presentation of the "capsules" dealing with algebra in the Thirty-first Yearbook of the National Council of Teachers of Mathematics, *Historical Topics for the Mathematics Classroom*. Paperback publication makes this material available in an economical and flexible form for use in algebra classrooms or by individuals whose mathematical interest at the moment is primarily in algebra.

"What is new today becomes old tomorrow," even in mathematics. The "new math" of a few years ago is now commonplace in many elementary and secondary schools across the country. Of course such terms as "new math" and "traditional math" still carry meaning for those professionally involved in the teaching of mathematics, although these terms may not carry exactly the same meaning for all persons. But such designations are related to chronological intervals and conceptual patterns that encompass only a small part of the overall history and development of mathematics.

The Thirty-first Yearbook of the NCTM is a constant reminder to its readers that mathematics does indeed have a history and that there are values to be derived from using some of this history in the present-day classroom. As stated in its preface, the primary objective of that yearbook is "to make available to mathematics classes important material from the history and development of mathematics with the hope that this will increase the interest of students in mathematics and their appreciation for the cultural aspects of the subject."

In the "overview" of the history of algebra given in the yearbook, John K. Baumgart states: "Although originally 'algebra' referred to equations, the word today has a much broader meaning, and a satisfactory definition requires a two-phase approach: (1) Early (elementary) algebra is the study of equations and methods for solving them. (2) Modern (abstract) algebra is the study of mathematical structures such as groups, rings, and fields—to mention only a few. Indeed, it is convenient to trace the development of algebra in terms

of these two phases, since the division is both chronological and conceptual."

The accompanying algebra capsules give brief sketches of some of the individual topics that are part of each of these developments. A glance at the table of contents readily reveals that no attempt has been made to separate them into these two groups nor to give them in any sort of chronological order. Included are such themes as the change from a geometric to an algebraic solution of problems, the development of algebraic symbolism, the algebraic contributions of different countries at different times, the origin and development of certain special topics in algebra, the search for generality and abstract structures.

While the capsule treatment has the advantage of permitting concentration on just one topic at a time, it also limits the discussion of the interrelationships with other algebraic topics and with other areas of mathematics as well. The overview of the history of algebra in the parent yearbook will be of help in giving the reader orientation and a general picture of some of these developments.

The Thirty-first Yearbook includes, in addition to the material on algebra, overviews and capsules on the history of numbers and numerals, computation, geometry, trigonometry, calculus, and modern mathematics, together with an essay on the history of mathematics as a teaching tool.

Those persons who contributed to the preparation of the entire project are acknowledged in the preface of the yearbook. Now thanks are expressed not only to them but to members of the Publications Committee of the NCTM for their encouragement and recommendation that this portion be made available separately as one of the Council's supplementary publications.

ARTHUR E. HALLERBERG
Chairman of the Editorial Panel
Thirty-first Yearbook of the NCTM

Contents

REFERENCE SYMBOLS AND BIBLIOGRAPHY

Cross-references within the text to related capsules are indicated by giving the number of the capsule in boldface type within square brackets. For capsules with numbers outside the interval from [**66**] through [**90**] the reader is referred to the Thirty-first Yearbook.

Complete bibliographical information for coded references appearing in the text within slashes or indicated under "For Further Reading" is given in the extensive bibliography of the yearbook.

The following selected bibliography, though brief, may serve two purposes here: to identify the more commonly cited references and to provide school libraries with a listing of books of general interest in the history of mathematics.

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Origin of the Word "Algebra"

from the yearbook overview by

JOHN K. BAUMGART

Exotic and intriguing is the origin of the word "algebra." It does not submit to a neat etymology as does, for example, the word "arithmetic," which is derived from the Greek *arithmos* ("number").

Algebra is a Latin variant of the Arabic word *al-jabr* (sometimes transliterated *al-jebr*) as employed in the title of a book, *Hisab al-jabr w'al-muqabalah,* written in Baghdad about A.D. 825 by the Arab mathematician Mohammed ibn-Musa al-Khowarizmi (Mohammed, son of Moses, the Khowarezmite). This treatise on algebra is commonly referred to, in shortened form, as *Al-jabr*.

A literal translation of the book's full title is "science of restoration (or reunion) and opposition," but a more mathematical phrasing would be "science of transposition and cancellation"—or, as Carl Boyer puts it /(g): 252-53/, "the transposition of subtracted terms to the other side of an equation" and "the cancellation of like [equal] terms on opposite sides of the equation." Thus, given the equation

$$x^2 + 5x + 4 = 4 - 2x + 5x^3,$$

al-jabr gives

$$x^2 + 7x + 4 = 4 + 5x^3,$$

and *al-muqabalah* gives

$$x^2 + 7x = 5x^3.$$

Perhaps the best translation would be simply "the science of equations."

While speaking of etymologies and al-Khowarizmi it is interesting to note that the word "algorism" (or algorithm), which means any

special process of calculating, is derived from the name of this same author, al-Khowarizmi, because he described processes for calculating with Hindu-Arabic numerals in a book whose Latin translation is usually referred to as *Liber algorismi* ("Book of al-Khowarizmi").

Perhaps a final philological comment on the lighter side is worthwhile. The Moroccan Arabs introduced the word *algebrista* ("restorer [that is, reuniter] of broken bones, bonesetter") into Moorish Spain. Since bonesetting and bloodletting were additional fringe benefits available at the barbershop, the local barber was known as an *algebrista*. Hence, also, the bloody barber poles!

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Capsule 66 Kenneth Cummins

EQUATIONS AND THE WAYS THEY WERE WRITTEN

IF A student of the time of Diophantus had been confronted with an expression of the now-common form illustrated by $x^2 - 7x + 12 = 0$, he would have been utterly baffled; this modern symbolic style is of relatively recent invention.

There is no complete agreement on "the time of Diophantus"; some authorities believe that he lived in the third century A.D., but some place him as early as the first century. It is known, however, that he was a Greek mathematician working "in residence" at the University of Alexandria, Egypt, and that he made a start on the use of algebraic symbolism, which eventually supplanted the writing of algebra in a verbal or prose style called "rhetorical algebra."

To illustrate rhetorical algebra we choose an example from an Arab mathematician of a later period: al-Khowarizmi, whose book *Al-jabr* (*c.* 825) both named and greatly influenced European algebra. (It is curious that even al-Khowarizmi used words for numbers, since it was his book *Liber algorismi* [to use the Latin] that introduced Hindu-Arabic numerals into Europe.) He states and solves, as follows, the problem given in modern notation as $x^2 + 21 = 10x$:

> What must be the amount of a square, which, when twenty-one dirhems are added to it, becomes equal to the equivalent of ten roots of that square? Solution: Halve the number of the roots; the half is five. Multiply this by itself; the product is twenty-five. Subtract from this the twenty-one which are connected with the square; the remainder is four. Extract its root; it is two. Subtract this from the half of the roots, which is five; the remainder is three. This is the root of the square which you required and the square is nine. Or you may add the root to the half of the roots; the sum is seven; this is the root of the square which you sought for, and the square itself is forty-nine.

Of course, his solution amounts to our writing

$$x = \frac{10}{2} \pm \sqrt{\left(\frac{10}{2}\right)^2 - 21} = 5 \pm \sqrt{4} = 3, 7.$$

If al-Khowarizmi's algebra seems prosaic, it might be worthwhile to comment that ideas often precede notation; symbolism is invented as needed.

"Syncopated algebra," the use of abbreviated words, was introduced by Diophantus; and somewhat later, in India, Brahmagupta (*c.* 628) invented his own system of abbreviations. Unfortunately, other writers often chose to ignore (or were unaware of) existing progress in notation; thus al-Khowarizmi used the rhetorical style of the preceding example.

The original of Diophantus' thirteen-volume work, the *Arithmetica*, has been lost, and the earliest existing copy of any part of the work was made more than a thousand years after it was written.

Here is an example from one of the earlier manuscripts, followed by interpretations in modern form and an explanation of the Greek:

$$\mathrm{K}^{\Upsilon}\beta \quad s\eta \pitchfork \Delta^{\Upsilon}\epsilon \quad \mathring{\mathrm{M}}\delta \quad \text{ἐστὶ} \quad \mu\delta;$$

that is,

$$x^3 2 \quad x8 - x^2 5 \quad 1 \cdot 4 \quad = \quad 44,$$

or

$$2x^3 + 8x - (5x^2 + 4) \quad = \quad 44.$$

K^{Υ} is an abbreviation for ΚΥΒΟΣ (*KUBOS*, "cube").
s is an abbreviation for αριθμος (*arithmos*, "number").
$\pitchfork$ is a combination of Λ and I in ΛΕΙΨΣΙΣ (*LEIPSIS*, "lacking").
Δ^{Υ} is an abbreviation for ΔΥΝΑΜΙΣ (*DUNAMIS*, "power").
$\mathring{\mathrm{M}}$ is an abbreviation for ΜΟΝΑΔΕΣ (*MONADES*, "units").

Equality is expressed by ἐστί ("is equal to") and also by ἰσ for ἴσος (*isos*, "equal").

The first nine letters of the Greek alphabet, α, β, γ, δ, ϵ, ς, ζ, η, and θ, stand for 1, 2, 3, 4, 5, 6, 7, 8, and 9; and ι, κ, λ, μ, ν, ξ, o, π, and ϙ (obsolete koppa), stand for 10, 20, 30, 40, 50, 60, 70, 80, and 90.

The example given above uses some capital letters, some lowercase. Later manuscripts use only lowercase letters.

To illustrate the syncopated style of Brahmagupta, we give the following example, with an interpretation into modern notation:

ya ka 7 *bha k(a)* 12 *ru* $\dot{8}$	$7xy + \sqrt{12} - 8$
ya v(a) 3 *ya* 10	$= 3x^2 + 10x.$

It will be quickly noted that equality is expressed by writing the left-hand member of the equation above the right-hand member (to use modern terminology). The shortened form *ya* stands for *yavattavat*, the first unknown; *ka* for *kalaka* ("black"), a second unknown; *bha* for *bhavita*, ("product"); *k(a)* for *karana*, ("irrational" or "root"). The dot placed above a number, as it is here placed over the 8, indicates a negative number; *ru* stands for *rupa*, ("pure" or "plain" number); *v(a)* for *varga*, ("square number"). Additional unknowns would have been expressed by using abbreviations for additional colors, thus: *ni* for *nilaca* ("blue"), *pi* for *pitaca* ("yellow"), *pa* for *pandu* ("white"), and *lo* for *lohita* ("red").

The accompanying list of examples will give some indication of the ways in which algebraic notation gradually progressed from the rhetorical stage to the symbolic. (See also the examples in the overview for this chapter and in [89].) To help the reader decipher some of the abbreviations we make the following brief introductory comments.

A pure number is often followed by *N*, *numeri*, or ϕ (analagous to our writing $7x^0$ for 7). Abbreviations for x are many, including *Pri.* for *primo* ("first"), *n°* for *numero* ("number," "unknown"), ρ for *res* ("thing," "unknown"), and *N* for *numerus* ("number," "unknown"). The square (of x) is written in many ways, including *Se.* for *secundo* ("second"). Addition and subtraction are often indicated by $\bar{p}$ for *piu* ("more") and $\bar{m}$ for *meno* ("less").

1494	Trouame .I.n°. che giõto al suo q̃drat° facia .12.
Pacioli	$x \quad + \quad x^2 \quad = \quad 12.$
1514	4Se. — 51Pri. — 30N dit is ghelijc 45.
Vander Hoecke	$4x^2 - 51x \quad - 30 \quad = \quad 45.$

1521 I ☐ e 32C° — 320 numeri.
Ghaligai $x^2 + 32x = 320.$

1525 Sit I𝔷 aequatus 12℞ — 36.
Rudolff $x^2 = 12x - 36.$

1545 cubus p̄ 6 rebus aequalis 20.
Cardano $x^3 + 6x = 20.$

1553 2 ℞ A + 2𝔷. aequata. 4335.
Stifel $2x\ A + 2x^2 = 4{,}335.$

1557 14.ꝛ. + .15.ꝗ ═══ 71.ꝗ.
Recorde $14x + 15 = 71.$

1559 I ◇ P 6ρ P 9 [I ◇ P 3ρ P 24.
Buteo $x^2 + 6x + 9 = x^2 + 3x + 24.$

1572 $\overset{6}{\smile}$I. p. $\overset{3}{\smile}$8. Eguale à 20.
Bombelli $x^6 + 8x^3 = 20.$

1585 3② + 4 egales à 2① + 4.
Stevin $3x^2 + 4 = 2x + 4.$

1591 I QC — 15 QQ + 85C — 225Q + 274N aequatur 120.
Viète $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x = 120.$

1631 $aaa - 3\ bba ═══ +2\ ccc.$
Harriot $x^3 - 3b^2x = 2c^3.$

1637 $yy \propto cy - \frac{cx}{b}y + ay - ac.$
Descartes

1693 $x^4 + bx^3 + cxx + dx + e = 0.$
Wallis

For Further Reading

CAJORI (d): I, 71–400
SANFORD (d) 153–59
D. E. SMITH (a): II, 421–35

Capsule 67 Wade Ellis

THE BINOMIAL THEOREM

The "arithmetic triangle" is often associated with the name of Blaise Pascal, who in 1653 discussed many of its properties and applied it to the expansion of $(a + b)^n$, with n a positive integer. He did not claim to have invented the "triangle" or the binomial theorem, but he was probably unaware that the Hindus and Arabs had worked with these ideas as early as the beginning of the twelfth century, when Omar Khayyam claimed to know the binomial expansion for degrees four, five, six, and higher (and for particulars referred the reader to another of his works—which has since been lost).

The Hindus and Arabs used the expansions of $(a + b)^2$ and of $(a + b)^3$ in finding square roots and cube roots. If they were given a positive number N and required to find its square root, they would choose a nearby perfect square number, say s^2, and let d be another number such that $s^2 + d = N$. The correction on s so that $(s + \text{correction})^2 = N$ was all that was required. Continuing to use modern notation here, we can describe their square-root process as follows: Let x be the required correction; then

$$N = s^2 + d = (s + x)^2 = s^2 + 2sx + x^2$$

$$d = x(2s + x)$$

$$x = \frac{d}{2s + x}.$$

By discarding the x on the right we obtain a first approximation to $\sqrt{N}$:

$$\sqrt{N} \approx s_1 = s + \frac{d}{2s}.$$

A machine program for this will be easy to write. Using $N = 5$ as an illustration, we have

$$5 = 2^2 + d = (2 + x)^2$$

$$x = \frac{d}{2s} = \frac{1}{2(2)} = 0.25$$

$$s_1 = 2 + 0.25 = 2.25.$$

Repeating this process, we have

$$5 = (2.25)^2 + d_1 = (2.25 + x_1)^2$$

$$x_1 = \frac{d_1}{2s_1} = \frac{-0.0625}{2(2.25)} = -0.01389$$

$$s_2 = s_1 + x_1 = 2.23611$$

$$d_2 = 5 - (2.23611)^2 = 0.00019;$$

and so on, until d_n (which does, in fact, converge to zero) is as close to zero as required.

For cube roots a similar iterative procedure results from letting $N = s^3 + d$. Then

$$x = \frac{d}{3s^2 + 3sx + x^2}$$

$$\approx \frac{d}{3s^2}.$$

It is interesting to observe that the correction term, x, which gave $d/2s$ and $d/3s^2$ for square and cube roots, reminds one of Newton's method in the calculus.

In 1676 Isaac Newton wrote two letters to Henry Oldenburg in which Newton stated without proof the binomial formula

$$(P + PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \cdots,$$

where A = first term $= P^{\frac{m}{n}}$, B = second term $= \frac{m}{n} AQ$, and so forth, and the exponent $\frac{m}{n}$ was a rational fraction (positive or negative). The form of the theorem more familiar to the modern reader is obtained if one makes the indicated substitution for $A, B, C, \ldots$:

$$(P + PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m}{n}-1}(PQ) + \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)}{1\cdot 2} P^{\frac{m}{n}-2}(PQ)^2 + \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{1\cdot 2\cdot 3} P^{\frac{m}{n}-3}(PQ)^3 + \cdots.$$

The first proof (not up to modern standards of rigor) for arbitrary positive integral power (i.e., m/n = positive integer) seems to be that given by Jakob (or Jacques) Bernoulli in his *Ars conjectandi,* which was published in 1713, eight years after his death. In 1826 the twenty-four-year-old Niels Henrik Abel, poverty-stricken and suffering from lumbar tuberculosis but already a famous mathematician, published the first general proof of the formula for arbitrary complex exponents. This appeared in the *Journal für die reine und angewandte Mathematik,* customarily referred to as *Crelle's Journal.*

It might be stated that in the expansion of $(1 + x)^\alpha$, the successive terms form a sequence that is finite only if α is a nonnegative integer. In case α is fractional or negative, the question of convergence—both of the sequence of successive terms and of the series, which is itself the general binomial expansion—immediately arises.

We do not often think of the binomial theorem, even in its general form, as opening doors to more advanced mathematics; yet a discussion of the two following expressions,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

and

$$\lim_{\substack{y\to 0\\ x \text{ fixed}}}(1+xy)^{1/y},$$

leads to the definition of the transcendental number e and the transcendental function e^x. With this in mind, it is no longer mysterious that the Maclaurin series expansion for e^x looks like a modification of the binomial expansion of $(1 + xy)^{1/y}$.

For Further Reading

ORE (b)

Capsule 68 *Elaine J. Tatham*

CONTINUED FRACTIONS

THE equality
$$\frac{318}{76} = 4 + \cfrac{1}{5 + \cfrac{1}{2 + \frac{1}{3}}}$$
shows that the common fraction 318/76 can be written as a *continued fraction.* If all the numerators in a continued fraction are 1's (as in the above example) it is called a *simple* continued fraction.

Perhaps the most interesting elementary property of continued fractions is their close relationship with the Euclidean algorithm for finding the greatest common divisor of two integers:

$$318 = 76(4) + 14$$
$$76 = 14(5) + 6$$
$$14 = 6(2) + 2$$
$$6 = 2(3) + 0$$

Remarks:
318 ÷ 76 gives a quotient of 4 and a remainder of 14, and so forth.
The last nonzero remainder, 2, is the G.C.D. of 318 and 76.

$$\frac{318}{76} = 4 + \cfrac{1}{5 + \cfrac{1}{2 + \frac{1}{3}}}$$

Remark:
To obtain the above, write
$$\frac{318}{76} = 4 + \cfrac{1}{\frac{76}{14}}$$
and then replace $\frac{76}{14}$ by $5 + \cfrac{1}{\frac{14}{6}}$, and so forth.

The striking similarity of the expressions in the parallel columns above (especially with respect to the digits 4, 5, 2, and 3) leads some writers to say that continued fractions were already known to the Greeks, "though not in our present notation."

Rafael Bombelli seems to have been the first to make explicit use of (infinite) continued fractions when he wrote the following in 1572 (modern notation is used here):

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \cdots}}};$$

and he probably recognized the above as a special case of

$$\sqrt{a^2 + b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cfrac{b}{2a + \cdots}}}.$$

The above expression for $\sqrt{13}$ is called an "infinite continued fraction" and can be obtained by equating it to $3 + 1/x$; then

$$\frac{1}{x} = \frac{\sqrt{13} - 3}{1} = \frac{4}{\sqrt{13} + 3} = \frac{4}{3 + \sqrt{13}} = \frac{4}{3 + \left(3 + \frac{1}{x}\right)} = \frac{4}{6 + \frac{1}{x}};$$

hence

$$\sqrt{13} = 3 + \frac{1}{x} = 3 + \frac{4}{6 + \frac{1}{x}};$$

now just keep replacing $1/x$ by $4/(6 + (1/x))$.

This process for finding an infinite sequence of successive approximations for $\sqrt{13}$ gives the first three convergents as follows:

$$C_1 = 3 \qquad\qquad = 3.$$

$$C_2 = 3 + \tfrac{4}{6} \qquad = 3\tfrac{2}{3}.$$

$$C_3 = 3 + \frac{4}{6 + \frac{4}{6}} = 3\tfrac{6}{10}.$$

These converge to $\sqrt{13}$, oscillating back and forth across $\sqrt{13}$ as shown in Figure [68]-1.

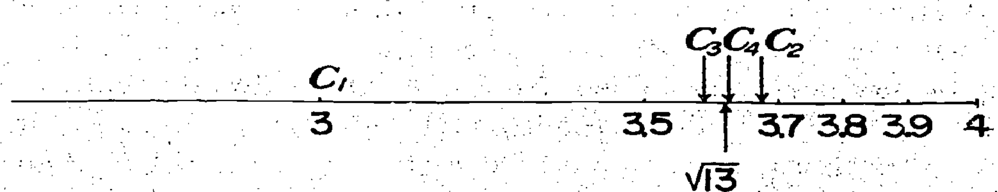

FIGURE [68]-1

John Wallis (*c.* 1685) found many properties of these convergents, including recurrence (or recursion) formulas that express a particular convergent, $C_k = N_k/D_k$, in terms of the preceding two sets of N's and D's. One of the interesting examples discussed by Wallis is the one discovered by William Brouncker (1658):

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

A modern form of symbolism was introduced by Christiaan Huygens (1629–1695), who expressed the ratio 77,708,431/2,640,858 in this form:

$$29 + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{5} + \frac{1}{1} + \frac{1}{4} \cdots .$$

This ratio actually arose in the solution of a practical problem which Huygens attacked in 1680, in designing the toothed wheels of his planetarium. In 365 days the annual movement of the earth is 359°45′40″31‴, while that of Saturn is 12°13′34″18‴. Converting to units of sixtieths of a second, 77,708,431 is to 2,640,858 as the period of Saturn is to the period of time during which the earth makes its revolution around the sun. The corresponding simple continued fraction given above is sometimes expressed today in the more convenient form (29; 2, 2, 1, 5, 1, 4, . . .), introduced by Dirichlet in 1854.

Huygens wished to find two smaller integers with almost the same ratio, so that no pair of smaller integers would yield a closer approximation. Denoting the simple continued fraction in the modern form $(a_0; a_1, a_2, a_3, \ldots)$, Huygens' approximation was made by attempting to determine a_k so that both $|a_k - a_{k+1}|$ and $|a_k - a_{k-1}|$ were maximized. He then used $(a_0; a_1, a_2, \ldots a_{k-1})$ as his approximation. Hence he chose $(29; 2, 2, 1) = 206/7$; his wheel of Saturn had 206 teeth while its motor wheel had 7 teeth. Using these numbers made it necessary to advance the wheel of Saturn by one tooth every 1,346 years.

It was Pietro Cataldi (1613) who began working on the theory of continued fractions and who also introduced—in his treatise published in Bologna on finding the square roots of numbers—the motivation for the notation that was to be used later by Huygens.

Leonhard Euler (1737) secured the foundation of the modern theory and showed that any quadratic irrational (like $\sqrt{13}$, above) can be represented by a simple repeating (or periodic) continued fraction; thus $\sqrt{13}$ can also be written in the following form:

$$3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}} \;.$$

More compactly, $\sqrt{13} = 3;11116\ 11116 \ldots.$

Johann Heinrich Lambert (1761) showed that the following simple continued fraction for π,

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}},$$

was not periodic and hence not a quadratic irrational ($a + \sqrt{b}$, a, b rational).

Joseph Louis Lagrange (1798) proved that periodic simple continued fractions represent solutions of quadratic equations with rational coefficients. Thus $\sqrt{13} - 1 = 2;\ 11116\ 11116 \ldots$ is a root of $x^2 + 2x - 12 = 0$. Lagrange also gave the first complete exposition of convergence of convergents. He showed that in general (see Fig. [68]-1) every odd convergent is less than all following convergents (in the sequence $C_1, C_2, C_3, C_4, C_5, \ldots$) and every even convergent is greater than all following convergents. From this (and the fact that the C's approach $\sqrt{13}$) it follows that, for example, C_4 differs from $\sqrt{13}$ by less than $(1/2)\cdot|C_3 - C_2|$.

Adrien Marie Legendre (1794) proved that every infinite continued fraction is irrational.

Thomas Joannes Stieltjes (1894) found a relationship between divergent series and convergent continued fractions which made it possible to define integration for the series; Stieltjes' integrals were to some extent a result of his work with continued fractions.

For Further Reading

BELL (a): 298–99, 476–78
CAJORI (d): II, 48–57
COURANT and ROBBINS: 49–51, 301–3
DANTZIG (b): 155–57, 312–16
NIVEN (a): 51–67
D. E. SMITH (a): II, 418–21
——— (c): I, 80–84
STRUIK (e): 111–15

Capsule 69 *Richard M. Park*

OUGHTRED AND THE SLIDE RULE

WILLIAM Oughtred (1574–1660), the vicar of Shalford and rector of Albury, Surrey, was one of the most influential mathematicians of his time. He was in great demand as a teacher, since the universities of that time offered little instruction in mathematics. A systematic treatment of much of the then-known work in arithmetic and algebra was published in his *Clavis mathematicae* (1631), which ran through six editions.

Oughtred placed unusual emphasis on mathematical symbols, developing or fostering many symbols in use today. Major examples are × for multiplication, :: for proportion, and — for difference.

Today, however, Oughtred is best remembered for his invention of both the circular and the rectilinear slide rules. His circular slide rule is described in his *Circles of Proportion* (1632) as eight fixed circles on one side of the instrument with an index operating much like a compass (Fig. [69]-1). Calling the outermost (largest) circle the first and the innermost circle the eighth, the scales on each of the eight circles are as shown below.

1. Sines from 5°45′ to 90°
2. Tangents from 5°45′ to 45°
3. Tangents from 45° to 84°15′
4. Logarithmically spaced integers 2, 3, 4, 5, 6, 7, 8, 9, 1
5. Equally spaced integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 0
6. Tangents from 84° to 89°24′

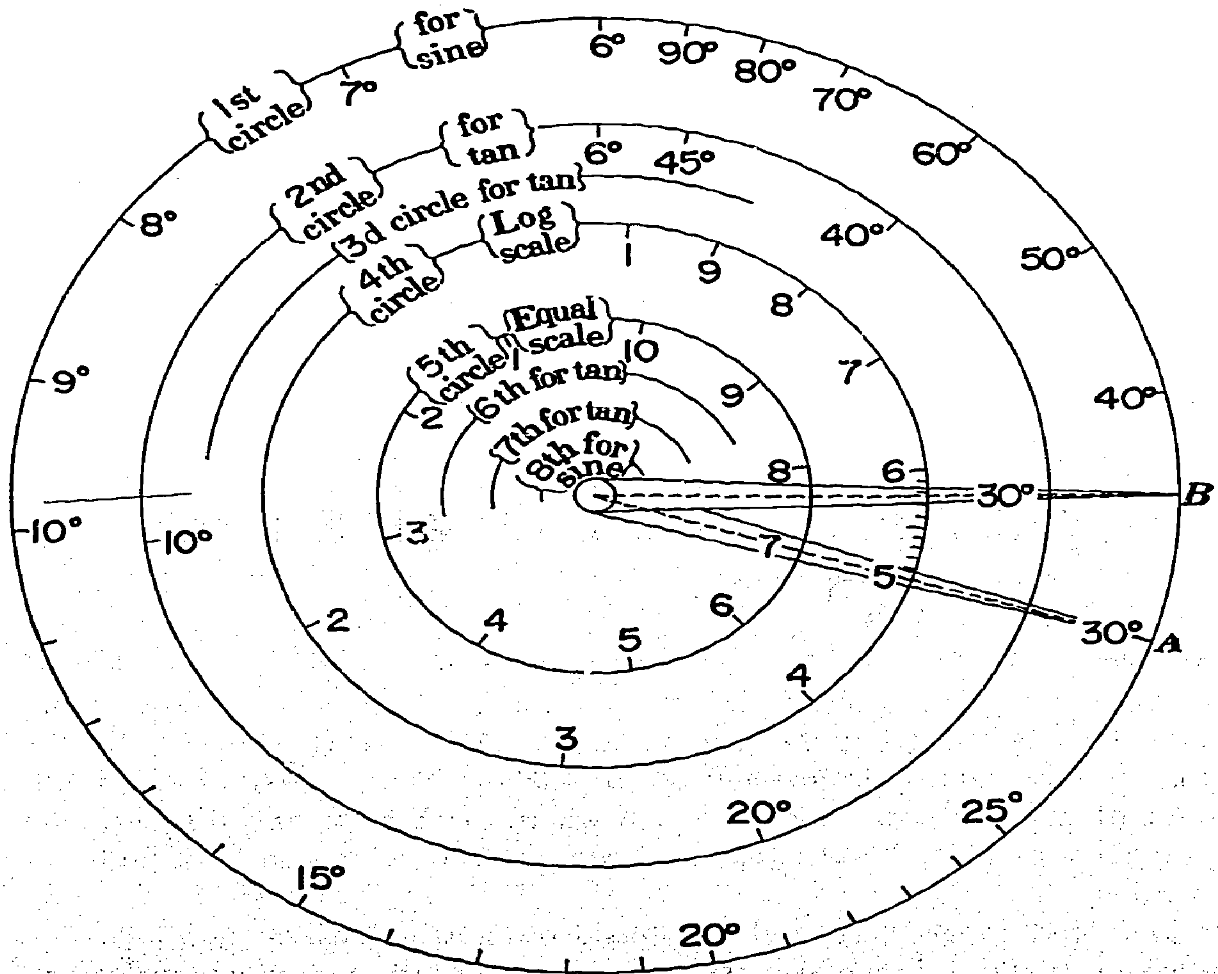

FIGURE [69]-1

7. Tangents from 35′ to 6°
8. Sines from 35′ to 6°

An example of its use: To find the value for sin 30° (see point *A* in Fig. [69]-1) one leg of the index (or compass) is placed at 30° on the first circle; the corresponding number on the fourth circle, 5, gives 0.5000 as the sine of 30°. Similarly, to find tan 30° refer to point *B*, 30° on the second circle, and read the corresponding answer, 0.5774, on the fourth circle. (Oughtred could get accuracy to four places.)

The fourth circle is used for multiplication. For $2 \cdot 3$ (see Figs. [69]-2 and -3) open and turn the two legs of the index so that they point to 1 and 2; then, with the angle α between the two legs held

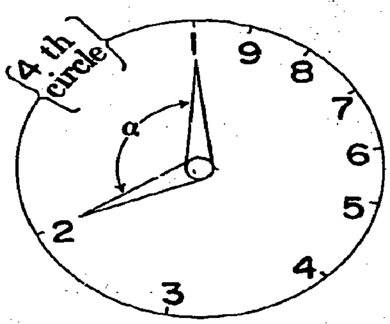

FIGURE [69]-2

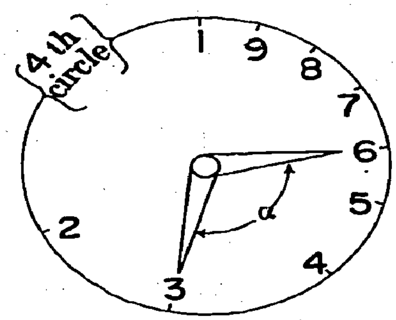

FIGURE [69]-3

constant, rotate the index so that one leg points to **3**. Then the other leg points to **6**, the desired product.

Around 1622 Oughtred invented his rectilinear slide rule, which consisted of two logarithmically calibrated rulers, one sliding along the other (without a fixed track or groove). He gave full credit to Edmund Gunter for the latter's invention in 1620 of a single rectilinear logarithmic scale, used to multiply numbers by adding the corresponding segments mechanically with the aid of a pair of dividers.

In 1630, two years before Oughtred published his *Circles of Proportion,* one of his former students, Richard Delamain, published *Grammelogia*. This, also, contained a description of a circular slide rule. Each man accused the other of having stolen his invention, but Cajori /(e): 158/ and D. E. Smith /(c): I, 160/ think it probable that each man invented the circular slide rule independently.

For Further Reading

CAJORI (c)
——— (d): 187–99
——— (e): 158–59

SANFORD (d): 343–47
D. E. SMITH (c): I, 160–64

Capsule 70 *Richard M. Park*

HORNER'S METHOD

WHAT we know today as Horner's method (for approximating real roots of polynomial equations with real, numerical coefficients) was known in an equivalent form by the Chinese for many centuries before it was published by Chhin Chiu-shao in 1247. It was called the "celestial element method"; and it appears also, though in more primitive form, in the *Nine Chapters,* written before the Christian era.

It is quite likely that in his travels Fibonacci (Leonardo of Pisa) learned of this method, which in 1225 he described rather well up to a certain point, after which he stopped explaining the method and merely gave the answer, to an excellent degree of accuracy. To solve (we use modern notation here)

$$x^3 + 2x^2 + 10x = 20,$$

he writes the equation in the form

$$x + \tfrac{1}{10}x^3 + \tfrac{1}{5}x^2 = 2,$$

from which it is clear that $x < 2$. The original equation shows that $x > 1$, since $1 + 2 + 10 < 20$. Then he shows that x cannot equal a rational fraction, a/b, because $(a/b) + (a^3/10b^3) + (a^2/5b^2)$ is not an integer; hence x is irrational. Further, x is not the square root of a positive integer, a, because the given equation implies that

$$x = \frac{20 - 2x^2}{10 + x^2},$$

which for $x = \sqrt{a}$ becomes the impossible statement that

$$\sqrt{a} = \frac{20 - 2a}{10 + a}.$$

Then Fibonacci abruptly gives the answer (in base sixty) as

$$x = 1^\circ 22' 7'' 42''' 33^{iv} 4^{v} 40^{vi},$$

that is, as

$$1+\frac{22}{60}+\frac{7}{60^2}+\frac{42}{60^3}+\frac{33}{60^4}+\frac{4}{60^5}+\frac{40}{60^6},$$

as it had probably been given to him during his travels.

François Viète (1600), apparently unaware of earlier results, gave a systematic process that showed a new insight into the general theory of equations, but the process becomes very laborious for equations of high degree.

Isaac Newton (1669) simplified Viète's method, the simplification being essentially like that found in texts in college algebra or theory of equations (not the Newton's method found in books on the calculus).

Paolo Ruffini (1803) and William George Horner (1819) independently worked out and published very similar methods for finding approximations of real roots of numerical polynomial equations. They both thought of their methods as better ways to find cube roots, fourth roots, and so on. At first they explained their methods in terms of the calculus, but later each of them was able to use only elementary algebra.

Ruffini's later method is actually closer than is Horner's to what present-day texts call "Horner's method."

Although Horner did not attend a university, he became a master in the Kingswood School of Bristol at the age of nineteen. He was not, however, a great mathematician. It was a stroke of good fortune that this mathematical accomplishment—his only one—was published in the *Philosophical Transactions* of the Royal Society (although not without some objections because of the elementary nature of his paper); the intricate style of his exposition made the work seem more impressive than it really was.

For Further Reading

Bell (a): 108–14
Coolidge (c): 186–94
D. E. Smith (a): II, 471–72
——— (c): 232–52

SOLUTION OF POLYNOMIAL EQUATIONS OF THIRD AND HIGHER DEGREES

THE first records of man's interest in cubic equations date from the time of the Old Babylonian civilization, about 1800–1600 B.C. Among the mathematical materials that survive are tables of cubes and cube roots, as well as tables of values of $n^2 + n^3$. Such tables could be used to solve cubics of special types.

For example, to solve the equation $2x^3 + 3x^2 = 540$, the Babylonians might have first multiplied by 4 and made the substitution $y = 2x$, giving $y^3 + 3y^2 = 2{,}160$. Letting $y = 3z$, this becomes $z^3 + z^2 = 80$. From the tables, one solution is found to be $z = 4$, and hence 6 is a root of the original equation.

In the Greek period concern with volumes of geometrical solids led easily to problems that in modern form involve cubic equations. The well-known problem of duplicating the cube is essentially one of solving the equation $x^3 = 2$. This problem, impossible of solution by ruler and compasses alone, was solved in an ingenious manner by Archytas of Tarentum (*c.* 400 B.C.), using the intersections of a cone, a cylinder, and a degenerate torus (obtained by revolving a circle about its tangent) /GRAESSER/.

The well-known Persian poet and mathematician Omar Khayyam (A.D. 1100) advanced the study of the cubic by essentially Greek methods. He found solutions through the use of conics. It is typical of the state of algebra in his day that he distinguished thirteen special types of cubics that have positive roots. For example, he solved equations of the type $x^3 + b^2x = b^2c$ (where b and c are positive numbers) by finding intersections of the parabola $x^2 = by$ and the circle $y^2 = x(c - x)$, where the circle is tangent to the axis of the parabola at its vertex. The positive root of Omar Khayyam's equation is represented by the distance from the axis of the parabola to a point of intersection of the curves.

The next major advance was the algebraic solution of the cubic. This

discovery, a product of the Italian Renaissance, is surrounded by an atmosphere of mystery; the story is still not entirely clear /CARDANO: ix–xii; FELDMAN (a)/. The method appeared in print in 1545 in the *Ars magna* of Girolamo Cardano of Milan, a physician, astrologer, mathematician, prolific writer, and suspected heretic, altogether one of the most colorful figures of his time.

The method has gained currency as "Cardan's formula," Cardan being the English form of the name. According to Cardano himself, however, the credit is due to Scipione del Ferro, a professor of mathematics at the University of Bologna, who in 1515 discovered how to solve cubics of the type $x^3 + bx = c$. As was customary among mathematicians of that time, he kept his methods secret in order to use them for personal advantage in mathematical duels and tournaments. When he died in 1526, the only persons familiar with his work were a son-in-law and one of his students, Antonio Maria Fior of Venice.

In 1535 Fior challenged the prominent mathematician Niccolo Tartaglia of Brescia (then teaching in Venice) to a contest because Fior did not believe Tartaglia's claim of having found a solution for cubics of the type $x^3 + bx^2 = c$. A few days before the contest Tartaglia managed to discover also how to solve cubics of the type $x^3 + ax = c$, a discovery (so he relates) that came to him in a flash during the night of February 12/13, 1535. Needless to say, since Tartaglia could solve two types of cubics whereas Fior could solve only one type, Tartaglia won the contest.

Cardano, hearing of Tartaglia's victory, was eager to learn his method. Tartaglia kept putting him off, however, and it was not until four years later that a meeting was arranged between them. At this meeting Tartaglia divulged his methods, swearing Cardano to secrecy and particularly forbidding him to publish it. This oath must have been galling to Cardano. On a visit to Bologna several years later he met Ferro's son-in-law and learned of Ferro's prior solution. Feeling, perhaps, that this knowledge released him from his oath to Tartaglia, Cardano published his version of the method in *Ars magna*. This action evoked bitter attack from Tartaglia, who claimed that he had been betrayed.

Although couched in geometrical language, the method itself is algebraic and the style syncopated. Cardano gives as an example the equation $x^3 + 6x = 20$ and seeks two unknown quantities, p and q, whose difference is the constant term 20 and whose product is the cube of 1/3 the coefficient of x, 8. A solution is then furnished by the difference of the cube roots of p and q. For this example the solution is

$$\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}.$$

This procedure easily applies to the general cubic after being transformed to remove the term in x^2.

This discovery left unanswered such questions as these: What should be done with negative and imaginary roots, and (a related question) do three roots always exist? What should be done (in the so-called irreducible case) when Cardano's method produced apparently imaginary expressions like

$$\sqrt[3]{81 + 30\sqrt{-3}} + \sqrt[3]{81 - 30\sqrt{-3}}$$

for the real root, -6, of the cubic $x^3 - 63x - 162 = 0$? These questions were not fully settled until 1732, when Leonhard Euler found a solution.

The general quartic equation yielded to methods of similar character; and its solution, also, appeared in *Ars magna*. Cardano's pupil Ludovico Ferrari was responsible for this result. Ferrari, while still in his teens (1540), solved a challenging problem that his teacher could not solve.

His solution can be described as follows: First reduce the general quartic to one in which the x^3 term is missing, then rearrange the terms and add a suitable quantity (with undetermined coefficient) to both sides so that the left-hand member is a perfect square. The undetermined coefficients are then determined so that the right-hand member is also a square, by requiring that its determinant be zero. This condition leads to a cubic, which can now be solved—the quartic can then be easily handled.

Later efforts to solve the quintic and other equations were foredoomed to failure, but not until the nineteenth century was this finally recognized. Carl Friedrich Gauss had proved in 1799 that every algebraic equation of degree n over the real field has a root (and hence n roots) in the complex field. The problem was to express these roots in terms of the coefficients by radicals. Paolo Ruffini, an Italian teacher of mathematics and medicine at Modena, is considered to have given (in 1813) an essentially satisfactory proof of the impossibility of doing this for equations of degree higher than four. Better known is the work of a brilliant young Norwegian mathematician, Niels Henrik Abel. After first thinking he had solved the general quintic, Abel found his error; and in 1824 he published at his own expense (in Christiania, now Oslo) his proof of its impossibility. His result appeared also, two years later, in the first volume of *Crelle's Journal* (Berlin), thus helping to inaugurate at a high level one of the

great mathematical periodicals of the world. Abel's work in turn stimulated the young Frenchman Évariste Galois (1811–1832), who before his early death in a duel showed that every equation could be associated with a characteristic group and that the properties of this group could be used to determine whether the equation could be solved by radicals.

For Further Reading

BOYER (g): 310–17
CARDANO: vii-xxii
COOLIDGE (c): 19–29
EVES (c)
 [3d ed. 217–21]
FELDMAN (a)
GRAESSER
MIDONICK: 583–98
NEUGEBAUER (a): 44, 51
ORE (a)
——— (b)
D. E. SMITH (a): II, 454–70
STRUIK (e): 62–73
JACOB YOUNG: 213–21

Capsule 72 Leonard E. Fuller

VECTORS

THE roots of vector algebra go back to the geometric concept of directed line segments in space. The composition of forces by the parallelogram law led to the idea of addition of vectors. Their representation as ordered sets of real numbers occurred only after the extension of number systems beyond the complex numbers.

Hermann Grassmann, in his monumental *Ausdehnungslehre*, published in 1844, freed his thinking from three-dimensional Euclidean space. He discussed manifolds of n dimensions and developed algebras for these systems. This enabled him to consider an extension of complex numbers to hypercomplex numbers. He made a significant stride when he found that he had to give up the property of commutativity of multiplication. This was the major stumbling block in the extension. His

work included also the theory of tensor calculus, which was destined to play a key role in the theory of relativity. Unfortunately, Grassmann's work was not properly understood by others, so its true significance had to wait for the passage of time. In 1862 he published a second edition of his work, in which he attempted to clarify the first and to add to it; but again he met with little success.

The year before Grassmann published the first edition of his *Ausdehnungslehre*, William Rowan Hamilton discovered the basic idea for quaternions. He, too, was bold enough to sacrifice the commutative property of multiplication. In 1853 he published *Lectures on Quaternions*, a work that was better understood and appreciated than Grassmann's, perhaps because it was not so general. Hamilton devoted the rest of his career to developing the theory of quaternions. He seemed convinced that this theory held the key to many ideas.

There was opposition to Hamilton's ideas, perhaps because of the complexity of the algebra involved. As a result, others tried to develop their own substitutes for it.

A disciple of Hamilton, Peter Guthrie Tait, devoted his life to quaternions. He stirred up a fight between mathematicians that extended over fifty years. His chief opponent was Josiah Willard Gibbs, who developed an excellent departure from quaternions with his vector analysis. A student of Gibbs, Edwin Bidwell Wilson, put the theory of vector analysis in book form in 1901. It is ironic that the idea that could have resolved the conflict much earlier was in Grassmann's *Ausdehnungslehre*. Actually, it was resolved by Grassmann's tensor calculus, which was further developed by C. G. Ricci, who published a work on it in 1888. At first, little attention was paid to this work; it was only after Einstein used it in his theory of relativity that it gained general acceptance. This theory of relativity vindicated the work of Grassmann and showed that he had been more than fifty years ahead in his thinking.

Today vectors are studied from the geometric point of view as directed line segments in three dimensions, largely as a result of Gibbs's work, and from the algebraic point of view as n-dimensional manifolds, largely as a result of Grassmann's.

For Further Reading

Bell (a): 182–211
Cajori (e): 334–45
Newman: I, 162–63, 697–98
D. E. Smith (c): II, 677–96

Capsule 73 *Leonard E. Fuller*

DETERMINANTS AND MATRICES

THE Japanese mathematician Seki Kowa (1683) systematized an old Chinese method of solving simultaneous linear equations whose coefficients were represented by calculating sticks—bamboo rods placed in squares on a table, with the positions of the different squares corresponding to the coefficients. In the process of working out his system, Kowa rearranged the rods in a way similar to that used in our simplification of determinants; thus it is thought that he had the *idea* of a determinant.

Ten years later in Europe Gottfried Wilhelm von Leibniz formally originated determinants and gave a written notation for them. In a letter to Marquis de L'Hospital Leibniz gave a discussion of a system of three linear equations in two unknowns /D. E. SMITH (c): I, 268–69/. A translation appears in the left-hand column, below, with a more modern version in the right-hand column.

I suppose that

$$10 + 11x + 12y = 0$$
$$20 + 21x + 22y = 0$$
$$30 + 31x + 32y = 0$$

$$a_{10} + a_{11}x + a_{12}y = 0.$$
$$a_{20} + a_{21}x + a_{22}y = 0. \qquad (1)$$
$$a_{30} + a_{31}x + a_{32}y = 0.$$

where . . . eliminating y first from the first and second equations, we shall have

$$\begin{matrix} 10.22 + 11.22x \\ -12.20 - 12.21\ldots \end{matrix} = 0,$$

$$(a_{10}a_{22} - a_{12}a_{20}) + (a_{11}a_{22} - a_{12}a_{21})x = 0.$$

and from the first and third

$$\begin{matrix} 10.32 + 11.32x \\ -12.30 - 12.31\ldots \end{matrix} = 0.$$

$$(a_{10}a_{32} - a_{12}a_{30}) + (a_{11}a_{32} - a_{12}a_{31})x = 0.$$

It remains now to eliminate the letter x . . . and as the result we shall have

$$\begin{array}{lcl} 1_0.2_1.3_2 & & 1_0.2_2.3_1 \\ 1_1.2_2.3_0 & = & 1_1.2_0.3_2. \\ 1_2.2_0.3_1 & & 1_2.2_1.3_0 \end{array}$$

$$\begin{array}{lcl} a_{10}a_{21}a_{32} & & a_{10}a_{22}a_{31} \\ +a_{11}a_{22}a_{30} & = & +a_{11}a_{20}a_{32} \\ +a_{12}a_{20}a_{31} & & +a_{12}a_{21}a_{30} \end{array}$$

or, moving all terms to the left side of the equation,

$$\begin{array}{l} a_{10}a_{21}a_{32} - a_{10}a_{22}a_{31} \\ +a_{11}a_{22}a_{30} - a_{11}a_{20}a_{32} = 0, \\ +a_{12}a_{20}a_{31} - a_{12}a_{21}a_{30} \end{array}$$

or

$$\begin{vmatrix} a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \\ a_{30} & a_{31} & a_{32} \end{vmatrix} = 0. \qquad (2)$$

(The reader may recall, or easily verify, that (2) is the condition for the three straight lines represented by (1) to pass through a common point.) The now-standard "vertical line notation" used in (2) above was given in 1841 by Arthur Cayley.

Determinants were invented independently by Gabriel Cramer, whose now well-known rule for solving linear systems was published in 1750, although not in present-day notation.

Many other mathematicians also made contributions to determinant theory—among them Alexandre Théophile Vandermonde, Pierre Simon Laplace, Josef Maria Wronski, and Augustin Louis Cauchy. It is Cauchy who applied the word "determinant" to the subject; in 1812 he introduced the multiplication theorem.

Although the idea of a matrix was implicit in the quaternions (4-tuples) of William Rowan Hamilton and also in the "extended magnitudes" (n-tuples) of Hermann Grassmann [72], the credit for inventing matrices is usually given to Cayley, with a date of 1857, even though Hamilton obtained one or two isolated results in 1852. Cayley says that he got the idea of a matrix "either directly from that of a determinant; or as a convenient mode of expression of the equations $x' = ax + by$, $y' = cx + dy$."

It was shown by Hamilton in his theory of quaternions [77] that one could have a logical system in which the multiplication is not commutative. This result was undoubtedly of great help to Cayley in working out his matrix calculus because matrix multiplication, also, is noncommutative.

Cayley's theory of matrices grew out of his interest in linear transformations and algebraic invariants, an interest he shared with James Joseph Sylvester. They investigated algebraic expressions that remained invariant (unchanged except, possibly, for a constant factor) when the variables were transformed by substitutions representing translations, rotations, dilatations ("stretchings" from the origin), reflections about an axis, and so forth. Thus, for example, if one transforms the conic

$$Ax^2 + Bxy + Cy^2 = K \tag{1}$$

by applying the substitution

$$x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$$

$$y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y',$$

which is a linear transformation representing a rotation of axes through 45°, this becomes

$$A'x'^2 + B'x'y' + C'y'^2 = K, \tag{2}$$

where

$$A' = -A + C, \qquad B' = \tfrac{1}{2}(A + B + C), \qquad C' = \tfrac{1}{2}(A - B + C).$$

It is easily checked that the "discriminant" $B^2 - 4AC$ of (1) is equal to the discriminant $B'^2 - 4\,A'C'$ of (2), no matter what values are used for A, B, C. Hence this discriminant, $B^2 - 4AC$, is called an invariant (under the rotation). Under the 45° rotation, $3x^2 + 2xy + 3y^2 = 5$ becomes $4x'^2 + 0x'y' + 2y'^2 = 5$. The discriminants are, respectively, $2^2 - 4 \cdot 3 \cdot 3$ and $0^2 - 4 \cdot 4 \cdot 2$ (both equal to -32).

Today, matrix theory is usually considered part of the broader subject of linear algebra, and it is a mathematical tool of the social scientist, geneticist, statistician, engineer, and physical scientist.

For Further Reading

BELL (a): 182–89, 424–27
——— (d): 378–405
CAJORI (e): 332–45
FELDMAN (b)
MIDONICK: 196–211
NEWMAN: I, 341–65

Capsule 74 *Anice Seybold*

BOOLEAN ALGEBRA

THE idea of laying down postulates for the manipulation of abstract symbols (not necessarily numbers) seems to have occurred first in England and at about the time of George Boole (1815–1864). Boole published his basic ideas in 1847 in a pamphlet entitled *Mathematical Analysis of Logic*. In 1854, in *An Investigation of the Laws of Thought on Which Are Founded the Mathematical Theories of Logic and Probabilities*, he presented a more thorough exposition of his work. "Boolean algebra" is a term often applied to the algebra of sets, although it can also be interpreted so as to yield what we now call "the propositional calculus" or "truth-function logic," which is studied largely by means of truth tables.

Boole used lowercase letters such as x, y, z, to denote sets, whereas we often use uppercase A, B, C, and so on. It is assumed that we can tell whether a given thing does or does not belong to a given set. A set can be described by saying it consists of all items having a given property or characteristic. The set containing no elements is called the null set—in Boole's notation written as the number 0, in modern notation written as $\emptyset$ or O, uppercase letter oh. The set of all elements under consideration (containing all sets under consideration and perhaps more, too) is the universal set—1 in Boole's notation and now frequently I, uppercase letter eye. If we take the set of all human beings for the universal set, then all human males, all people over fifty years old, all blue-eyed people, and all brown-eyed people are four differents sets that are subsets of the universal one. The set of all two-headed people is the null set (we hope).

Sets can be combined to form new sets in two basic ways. The logical

product or intersection of two sets x and y (or A and B)—denoted by Boole as xy or $x \cdot y$ (now frequently as $A \cap B$, called "A cap B")—consists of all elements that are in both sets. If A is the set of all human males and B is the set of all blue-eyed people, then $A \cap B$ is the set of all blue-eyed men. If C is the set of all brown-eyed people, then we have $B \cap C = \emptyset$, where $=$ is used to connect two different symbols for the same set (in this case, the null set). By the meaning of logical product we must have $X \cap X = X$. Boole wrote this as $x^2 = x$. When this equation is regarded as a condition on unknown numbers rather than as a set-theoretic statement, it has only 0 and 1 as roots. This led Boole to search out his set-theoretic interpretations for 0 and 1 which we have already observed.

By logical sum of two sets A and B—denoted $A \cup B$ and called "A cup B"—we mean the set whose members are members of the set A or the set B or both. Using A and B as in the last paragraph, the set $A \cup B$ would consist of all people who are males or who have blue eyes, including, of course, all blue-eyed men.

Boole's "logical sum" was a little different. His logical sum of sets x and y—denoted $x + y$, read "x plus y"—consisted of elements in x or y but not in both. Just as we agreed with Boole that $x^2 = x$, we might have expected him to agree with us that $1 + 1 = 1$ and $x + x = x$. But his logical sum $x + x$ is difficult to interpret. Whenever it occurred he gave it the formal designation $2x$; this caused him complications that need not concern us here.

The analogues of certain laws in ordinary algebra are seen to hold in Boolean algebra. For instance, $A \cap B = B \cap A$ is the commutative law for logical products. Also,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

is an analogue of the distributive law. The correspondence of this law to the distributive law of ordinary algebra is especially obvious when we use Boole's symbols:

$$x(y + z) = xy + xz.$$

Another way of constructing a new set comes, not from combining two sets, but from complementation. If we remove from the universal set I all members of the set A, the remaining elements constitute a set called the complement of A and variously denoted by $I - A$, $-A$, A', and $\bar{A}$. By definition, $A \cup A' = I$, and $A \cap A' = 0$.

In the propositional calculus, letters stand for statements that may be true or false instead of for numbers (as in high school and college

algebra) or for sets (as in Boolean algebra). To give some indication of the relation between Boolean algebra and the propositional calculus, we mention only that if A is "Roses are red" and B is "Violets are blue," then $A \wedge B$ is "Roses are red and violets are blue," and $A \vee B$ is "Either roses are red or violets are blue or both statements are true."

John Venn (1834–1923), a contemporary of Boole's and also an Englishman, invented a way of representing clearly such Boolean expressions as the right and left members of the distributive law. Similar diagrams had been invented independently by Leonhard Euler (these were called Euler circles) and by Augustus De Morgan and others /SISTER STEPHANIE/. In Venn diagrams we draw a fence around all members of a set so as to exclude all nonmembers. Then the "area" common to the regions representing the two sets, the shaded area in Figure [74]-1, represents their logical product. Figure [74]-2 represents the case where $A \cap B = \emptyset$.

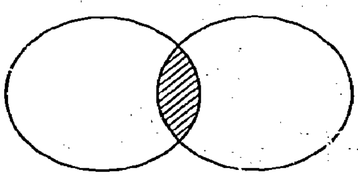

FIGURE [74]-1

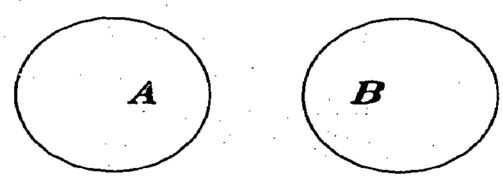

FIGURE [74]-2

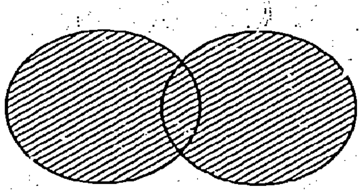

FIGURE [74]-3

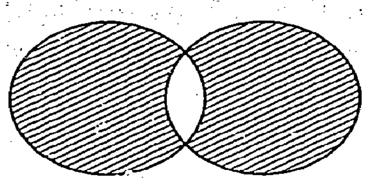

FIGURE [74]-4

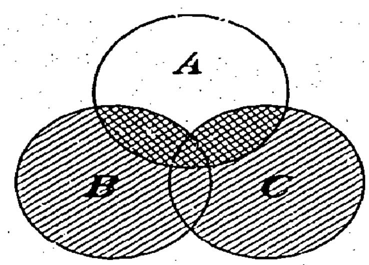

FIGURE [74]-5

The logical sum as defined by modern mathematicians would be represented by the shaded area in Figure [74]-3. However, according to Boole, the logical sum would be represented by the shaded area in Figure [74]-4. In order to find the Venn representation of $A \cap (B \cup C)$, the left member of our distributive law, we shade first the logical sum $B \cup C$, then its logical product with A, obtaining the doubly shaded area in Figure [74]-5.

Similar analysis of the right member of the distributive law yields the same set. Hence the two members are merely different names for the same set. (The reader might like to apply Venn diagrams to the

other distributive law of Boolean algebra, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.)

The most interesting recent development in connection with Boolean algebra is its application to the design of electronic computers through the interpretation of Boolean combinations of sets as switching circuits. The logical product of two sets corresponds to a circuit with two switches in series. Electricity flows in such a circuit only if both the first and second switches are closed. The logical sum of two sets corresponds to a circuit with two switches in parallel. Electricity flows in such a circuit if either one or the other or both switches are closed.

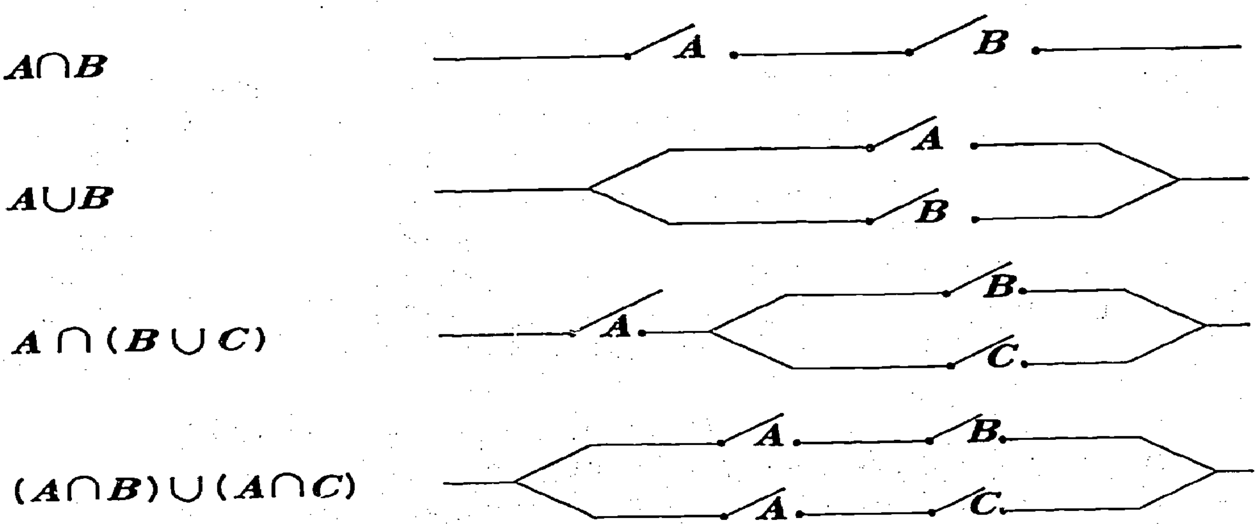

FIGURE [74]-6

In the last diagram of FIGURE [74]-6 the two A switches must be linked mechanically so that they are always both open or both closed. The last two circuits are equivalent (they correspond to identical sets by the distributive law); but the hardware for the first of these, $A \cap (B \cup C)$, is simpler.

For Further Reading

BELL (d): 433–47
BOOLE (a)
——— (b)
CAJORI (d): II, 290
MIDONICK: 147–65, 774–85
NEWMAN: III, 1852–1931
SISTER STEPHANIE

CONGRUENCE (Mod m)

LET m be a fixed, positive integer. For arbitrary integers x and y we write $x \equiv y \pmod{m}$, read "x is congruent to y, modulo m," in case the integer $x - y$ is divisible by the integer m. The concept and notation were introduced by Carl Friedrich Gauss in 1801, when he was twenty-four years old. The integer m is called the modulus.

The property described above means that there exists an integer q such that $x - y = qm$, or (what is the same) $x = y + qm$. For every integer x, the long-division process guarantees the existence of integers q and r such that $x = qm + r$, $0 \leq r < m$. Since x is thus congruent to r, modulo m, it follows that (modulo m) each integer x is congruent to one and only one of the integers $0, 1, \cdots, m - 1$; this integer is called the "least residue" of x, modulo m.

From the definition one can readily prove that—

1. If $x \equiv y \pmod{m}$, $y \equiv z \pmod{m}$, then $x \equiv z \pmod{m}$.
2. If $x \equiv y \pmod{m}$, then $y \equiv x \pmod{m}$.
3. If $x \equiv y \pmod{m}$, $a \equiv b \pmod{m}$, then
   a) $x + a \equiv y + b \pmod{m}$.
   b) $x - a \equiv y - b \pmod{m}$.
   c) $xa \equiv yb \pmod{m}$.
   d) $x^k \equiv y^k \pmod{m}$, k any positive integer.
   e) $kx \equiv ky \pmod{m}$, k any integer.

It follows that if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

where x and all the coefficients a_i are integers, and if $x \equiv y$ and every $a_i \equiv b_i$, modulo m, then

$$f(x) \equiv b_n y^n + \cdots + b_0 \pmod{m}.$$

Although congruences form a vital tool in the theory of integers, Gauss recognized their utility, also, in showing certain polynomial equations to have no rational roots. Consider the equation

$$f(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

where all a_i are integers. All rational roots of $f(x)$ are known to be integers dividing the constant term, a_0; call the integral divisors of a_0 "potential roots" of $f(x) = 0$.

If r is actually an integral root of $f(x) = 0$, then $f(r) = 0$, whence $f(r) \equiv 0 \pmod m$ for every choice of the modulus m. In considering a potential root r, if in some manner we find a positive integer m such that $f(r) \not\equiv 0 \pmod m$, then we are assured that r is not a root of $f(x) = 0$. The value of this method for eliminating potential roots lies in the fact that calculating $f(r)$ to determine whether $f(r) = 0$ is often far more difficult than "calculating $f(r)$, modulo m." The latter phrase refers to the determination of the least residue of $f(r)$, modulo m; if this residue is not 0, then $f(r) \not\equiv 0 \pmod m$ and $f(r) \neq 0$.

It is convenient to use the same modulus m in checking all potential roots r, but this is not essential. In selecting m, one will never gain any knowledge from an m that is a factor of a_0 and of r, for then we always find $f(r) \equiv 0 \pmod m$.

An example is shown below.

$$f(x) = x^{14} + x^3 - x^2 + x + 6 = 0.$$

Potential roots are ± 6, ± 3, ± 2, and ± 1. We try the modulus $m = 5$, since this is the smallest positive integer not dividing 6. Note that in any congruence modulo 5, the term 6 may be replaced by 1. Thus

$$f(1) \equiv 1 + 1 - 1 + 1 + 1 \equiv 3 \pmod 5$$

$$f(-1) \equiv 1 - 1 - 1 - 1 + 1 \equiv -1 \equiv 4 \pmod 5$$

$$f(2) \equiv 2^{14} + 3 - 4 + 2 + 1 \equiv 2^{14} + 2 \pmod 5.$$

Since

$$2^4 \equiv 1 \pmod 5,$$

it follows that

$$2^{14} \equiv 2^{10} \cdot 2^4 \equiv 2^{10} \equiv 2^6 \equiv 2^2 \pmod 5$$

$$f(2) \equiv 4 + 2 \equiv 1 \pmod 5$$

$$f(-2) \equiv 2^{14} - 8 - 4 - 2 + 1 \pmod 5$$

$$\equiv 4 - 3 - 4 - 2 + 1 \equiv 1 \pmod 5$$

$$f(3) \equiv 3^{14} + 2 - 4 + 3 + 1 \equiv 3^{14} + 2 \pmod 5.$$

Then

$$3^4 \equiv 9 \cdot 9 \equiv 4 \cdot 4 \equiv 1 \pmod 5$$

$$3^{14} \equiv 3^{10} \equiv 3^6 \equiv 3^2 \equiv 4 \pmod 5$$

$$f(3) \equiv 4 + 2 \equiv 1 \pmod 5$$

$$f(-3) \equiv 3^{14} - 2 - 4 - 3 + 1 \pmod 5$$

$$\equiv 3^{14} - 3 \equiv 4 - 3 \equiv 1 \pmod 5.$$

Since

$$6 \equiv 1 \pmod 5$$

it follows that

$$6^i \equiv 1^i \pmod 5;$$

$$f(6) \equiv f(1) \equiv 3 \pmod 5.$$

Similarly,

$$-6 \equiv -1 \pmod 5;$$

$$f(-6) \equiv f(-1) \equiv 4 \pmod 5.$$

In every case the least residue fails to be 0. Thus no potential root is an actual root, whence $f(x) = 0$ has no rational roots.

For Further Reading

MIDONICK: 380–86
ORE (c): 209–33
STRUIK (e): 49–54

Capsule 76 Eugene W. Hellmich

COMPLEX NUMBERS (THE STORY OF $\sqrt{-1}$)

HISTORY shows the necessity for the invention of new numbers in the orderly progress of civilization and in the evolution of mathematics.

The story of $\sqrt{-1}$, the imaginary unit, and of $x + yi$, the complex number, originates in the logical development of algebraic theory.

Deploring the use of the word "imaginary" by calling it "the great algebraic calamity" but "too well established for mathematicians to eradicate" is quite proper from the modern point of view; but the use of this word reflects the elusive nature of the concept for distinguished mathematicians who lived centuries ago.

Early consideration of the square root of a negative number brought unvarying rejection. It seemed obvious that a negative number is not a square, and hence it was concluded that such square roots had no meaning. This attitude prevailed for a long time.

Perhaps the earliest encounter with the square root of a negative number is in the expression $\sqrt{81 - 144}$, which appears in the *Stereometrica* of Heron of Alexandria (c. A.D. 50); the next known encounter is in Diophantus' attempt to solve the equation $336x^2 + 24 = 172x$ (as we would now write it), in whose solution the quantity $\sqrt{1{,}849 - 2{,}016}$ appears (again using modern notation).

The first clear statement of difficulty with the square root of a negative number was given in India by Mahavira (*c.* 850), who wrote: "As in the nature of things, a negative is not a square, it has no square root." Nicolas Chuquet (1484) and Luca Pacioli (1494) in Europe were among those who continued to reject imaginaries.

Girolamo Cardano (1545), who is also known as Jerome Cardan, is credited with some progress in introducing complex numbers in his solution of the cubic equation, even though he regarded them as "fictitious." He is credited also with the first use of the square root of a negative number in solving the now-famous problem, "Divide 10 into two parts such that the product $\cdots$ is 40," which Cardano first says is "manifestly impossible"; but then he goes on to say, in a properly adventurous spirit, "Nevertheless, we will operate." (This was due, no doubt, to his medical training!) Thus he found $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ and showed that they did indeed have a sum of 10 and a product of 40.

Cardano concludes by saying that these quantities are "truly sophisticated" and that to continue working with them would be "as subtle as it would be useless."

Cardano did not use the symbol $\sqrt{-15}$. His designation was "$R_x.\ m$," that is, "radix minus," for the square root of a negative number. Rafael Bombelli (*c.* 1550) used "*d.m*" for our $\sqrt{-1}$. Albert Girard (1629) included symbolism such as "$\sqrt{-2}$." René Descartes (1637) contributed the terms "real" and "imaginary." Leonhard Euler (1748) used "i" for $\sqrt{-1}$. Caspar Wessel (1797) used "$\sqrt{-1} = \epsilon$." Carl Friedrich

Gauss (1832) introduced the term "complex number." William Rowan Hamilton (1832) expressed the complex number in the form of a number-couple.

Bombelli continued Cardano's work. From the equation $x^2 + a = 0$, he spoke of "$+\sqrt{-a}$" and "$-\sqrt{-a}$." The special case of this equation, $x^2 + 1 = 0$, affords an excellent approach to i and i^2, as follows:

If $x^2 + 1 = 0$, then $x^2 = -1$ and $x = \pm\sqrt{-1}$. Now, if $i = \sqrt{-1}$, then $i^2 + 1 = 0$ when x is replaced by i, and $i^2 = -1$. From this it follows as a good exercise that $i^3 = -\sqrt{-1}$, $i^4 = 1$, $i^5 = \sqrt{-1}, \cdots$, $i^{243} = -\sqrt{-1}$, and so forth.

In his *Algebra* (1673, republished in 1693 in *Opera mathematica*; see /D. E. SMITH (c): I, 48/) John Wallis associated "—1600 square perches" with a loss and then supposed this to be in the form of a square with a side [160 square perches = 1 English acre]:

> What shall this side be? We cannot say it is 40, nor that it is -40. (Because either of these multiplied into itself, will make $+1600$; not -1600). But thus rather, that it is $\sqrt{-1600}$, (the Supposed Root of a Negative Square:) or (which is equivalent thereunto) $10\sqrt{-16}$, or $20\sqrt{-4}$, or $40\sqrt{-1}$.

Wallis, Wessel (1798), Jean Robert Argand (1806), Gauss (1813), and others made significant contributions to the understanding of complex numbers through graphical representation, and in 1831 Gauss defined complex numbers as ordered pairs of real numbers for which $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$, and so forth. Wessel's representation is given as follows /D. E. SMITH (c): I, 60/:

> Let $+1$ designate the positive rectilinear unit and $+\epsilon$ a certain other unit perpendicular to the positive unit and having the same origin; then the direction angle of $+1$ will be equal to 0°, that of -1 to 180°, that of $+\epsilon$ to 90°, and that of $-\epsilon$ to $-90°$ or 270°. By the rule that the direction angle of the product shall equal the sum of the angles of the factors, we have: $(+1)(+1) = +1$; $(+1)(-1) = -1$; $(-1)(-1) = +1$; $(+1)(+\epsilon) = +\epsilon$; $(+1)(-\epsilon) = -\epsilon$; $(-1)(-\epsilon) = +\epsilon$; $(+\epsilon)(+\epsilon) = -1$; $(+\epsilon)(-\epsilon) = +1$; $(-\epsilon)(-\epsilon) = -1$. From this it is seen that ϵ is equal to $\sqrt{-1}$, and the divergence of the product is determined such that not any of the common rules of operation are contravened.

Of a similar representation it has been said /BELL (d): 234/:

> All this of course proves nothing. *There is nothing to be proved*; we *assign*

to the symbols and operations of algebra *any meanings whatever* that will lead to consistency. Although the *interpretation* ··· *proves* nothing, it may suggest that there is no occasion for anyone to muddle himself into a state of mystic wonderment over nothing about the grossly misnamed "imaginaries."

A geometric representation credited to Wessel and Argand independently is based on the geometric principle that the altitude to the hypotenuse of a right triangle is a mean proportional between the segments into which the altitude divides the hypotenuse. In Figure [76]-1, $OD_1 = d_1 = +1$, $OD_2 = d_2 = -1$. $\angle D_1RD_2$ is a right angle, and $OR = d$. Then $d_1 : d = d : d_2$. Now $d = \sqrt{d_1d_2} = \sqrt{+1 \cdot -1} = \sqrt{-1} = i$.

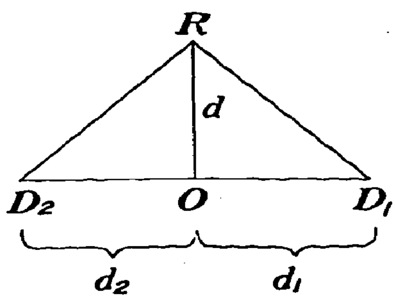

FIGURE [76]-1

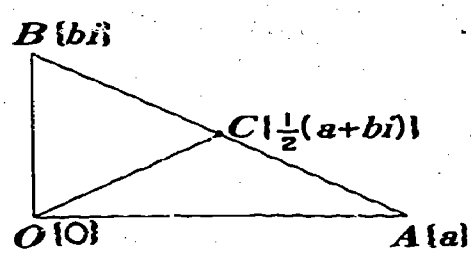

FIGURE [76]-2

Some interesting geometric proofs can result from the representation of the complex number $a + bi$ by the point in the plane with rectangular coordinates a and b. An example is the proof that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices. In Figure [76]-2 O is the vertex of the right angle of right triangle AOB, and C is the midpoint of the hypotenuse AB. Using the coordinates in the figure,

$$OC = |\tfrac{1}{2}(a + bi) - 0| = \tfrac{1}{2}|(a + bi)| = \tfrac{1}{2}\sqrt{a^2 + b^2},$$

and

$$AB = |a - bi| = \sqrt{a^2 + b^2}.$$

Hence

$$OC = \tfrac{1}{2}\,AB = BC = CA.$$

Lastly, among the more valuable relations involving the imaginary, is that suggested by Abraham De Moivre (1730):

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

An illustration of De Moivre's relation in the development of trigonometric identities follows /JONES (c)/:

(1) $$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

But by the binomial theorem we have

(2) $$(\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta.$$

Equating the right-hand members of (1) and (2), we have

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta).$$

Equating the real parts gives

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= 4\cos^3\theta - 3\cos\theta.$$

Equating the imaginary parts gives

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= -4\sin^3\theta + 3\sin\theta.$$

For Further Reading

BELL (d): 233–34
JONES (c)
MIDONICK: 804–14
D. E. SMITH (a): II, 261–67
——— (c): I, 46–66

Capsule 77 *Sam Perlis*

QUATERNIONS

VECTORS are objects that can be added or subtracted, and multiplied amongst themselves; they can also be multiplied by real numbers. In each case the result is another vector.

William Rowan Hamilton (who was born in Dublin in 1805 and was appointed professor of astronomy at Trinity College, Dublin, in 1827) was disturbed by the lack of any concept of *quotient* of vectors. That is, for any two vectors u and v, with $v \neq 0$, he wanted to find a unique vector q such that the vector product qv was equal to u. His investigations showed the system of vectors to be too small for this purpose and led him to an enlarged system whose members he called "quaternions." His work stirred up considerable disputation throughout the Western world on the question whether quaternions should replace vectors as an everyday tool in physics and mathematics, and it resulted in the formulation of an international association to study the question. We shall look briefly at the way in which Hamilton was led to quaternions.

Consider a rectangular coordinate system with axes X, Y, and Z and with unit vectors i, j, and k drawn on these respective axes. All vectors used herein emanate from the origin, whence the vector terminating in the point (x, y, z) is $xi + yj + zk$. Let u and v be two vectors; $u = ai + bj + ck$, and $v = di + ej + fk \neq 0$. We shall consider a process for converting v into u, meanwhile counting how many real numbers are needed to specify the process completely in the general case.

First, vectors v and u determine a plane π. Imagine a movable vector v_0, which initially lies on top of v. In the plane π we rotate v_0 until it lies on the ray containing vector u, the angle of rotation being designated as δ.

This number δ does not determine our rotation, since we are not content to rotate v_0 through *any* angle equal to δ, but only through an angle δ lying in the appropriate plane. We therefore consider what numbers may serve to specify the particular plane π through the origin. If a movable plane π_0 is pictured initially as coinciding with

the XZ plane, we may rotate π_0 about the Z-axis until it includes v; next, we rotate π_0 about v until it contains the vector u. In this final position the plane π_0 coincides with π; and if γ and β are the two angles employed, the triple (β, γ, δ) determines the contemplated rotation carrying the ray of vector v to that of u.

The length of v may very well differ from that of u, so we now multiply the former length by some constant α to convert it to that of u. Altogether four constants, α, β, γ, and δ, serve to convert v to u. To express this "fourness" Hamilton coined the name "quaternion" for whatever algebraic object he could find to accomplish the desired conversion.

It turned out that his purposes were served admirably by the notation

(1) $$w = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k,$$

where α_0, α_1, α_2, and α_3 are arbitrary real numbers. These symbols are to be combined under addition and subtraction by the usual rules. For example, if w' is given by

(2) $$w' = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k,$$

then both $w + w'$ and $w' + w$ are equal to

$$(\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)i + (\alpha_2 + \beta_2)j + (\alpha_3 + \beta_3)k.$$

The product ww' is defined by use of the usual distributive laws of algebra together with the following stipulations: $ij = k$; $ji = -k$; $jk = i$; $kj = -i$; $ki = j$; $ik = -j$; and $i^2 = j^2 = k^2 = -1$. Thus, for

(3) $$w = 1 + 2i + 3j + 4k; \qquad w' = 2 + i + 5k,$$

we find that

$$\begin{aligned} ww' &= 2 + 4i + 6j + 8k \\ &\quad + i - 2 + 3ji + 4ki \\ &\quad + 5k + 10ik + 15jk - 20 \\ &= -20 + 20i + 10k. \end{aligned}$$

A similar computation shows that $w'w = -20 - 10i + 12j + 16k$.

Since $ww' \neq w'w$ in the computations above, the commutative law of multiplication is not valid for quaternions. Another instance is given by the equations $ij = k$, $ji = -k$. For special pairs w and w', however, the product may be commutative. This is the case, for ex-

ample, if w is arbitrary and $w' = \beta_0 + 0i + 0j + 0k = \beta_0$; $ww' = w'w = \beta_0 w$.

The system of quaternions so constructed includes the familiar vectors $ai + bj + ck$; and when the laws of quaternion addition and multiplication are applied to these vectors, the usual results are obtained except that products now consist of a real term plus the usual vector product. But now we shall demonstrate that an additional feature is present: Every nonzero vector—also every nonzero quaternion —has an inverse in the system of quaternions.

The quaternion w in (1) is 0 if and only if all of its coefficients are 0. Let $w \neq 0$, whence the number

$$p = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \tag{4}$$

is a real, positive number. If we write w as

$$w = \alpha_0 + v, \qquad v = \alpha_1 i + \alpha_2 j + \alpha_3 k,$$

then v is called the "vector part" of w; and $\bar{w} = \alpha_0 - v$ is called the "conjugate" of w. Note that the conjugate of $\bar{w}$ is w. The norm of w is defined to be $w\bar{w}$. A short computation shows that both w and $\bar{w}$ have norm equal to the number p in (4): $w\bar{w} = p = \bar{w}w$. It follows that $w((1/p)\bar{w}) = 1 = ((1/p)\bar{w})w$, whence $(1/p)\bar{w}$ is the inverse of w. For the quaternion w in (3) the inverse is $(1/30)(1 - 2i - 3j - 4k)$.

As a standard device for everyday use in physics, quaternions have disappeared entirely. They are, however, very much alive now with a different raison d'être. Today mathematicians are interested in studying number systems in their entirety, in learning their properties, and in learning how to construct new ones. One prominent type is called an associative division algebra over a field. It is known that there are only three such algebras over the real field: (1) the real number system, (2) the complex number system, and (3) the system of quaternions. Thus the system of quaternions may be designated as the only noncommutative associative division algebra over the real field.

The noncommutativity of quaternion multiplication gives rise to a curious property. An equation of degree n can no longer be said to have at most n distinct roots, at least not if quaternion solutions are admitted. For example, the quadratic equation $w^2 + 1 = 0$ has three obvious quaternion solutions: $w = i$, $w = j$, and $w = k$. In actuality there are infinitely many. It is easy to verify that $w = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ satisfies the condition $w^2 + 1 = 0$ if and only if $\alpha_0 = 0$ and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$.

For Further Reading

Bell (a): 182–211
——— (d): 340–61
Boyer (g): 624–26
D. E. Smith (c): II, 677–83

Capsule 78 *Gertrude V. Pratt*

EARLY GREEK ALGEBRA

The algebra of the early Greeks (of the Pythagoreans and Euclid, Archimedes, and Apollonius, 500–200 B.C.) was geometric because of their *logical* difficulties with irrational and even fractional numbers and their *practical* difficulties with Greek numerals [4], which were somewhat similar to Roman numerals and just as clumsy. It was natural for the Greek mathematicians of this period to use a geometric style for which they had both taste and skill.

The Greeks of Euclid's day thought of the product ab (as we write it) as a rectangle of base b and altitude a, and they referred to it as "the rectangle contained by CD and DE" (Fig. [78]-1).

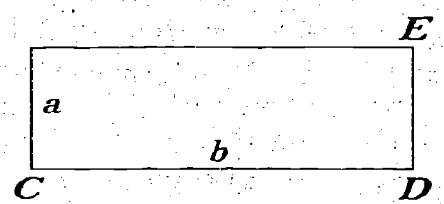

Figure [78]-1

To illustrate the style and method of Greek geometric algebra we show how they solved a particular kind of quadratic equation. The theorem—in this case, really a problem to be solved—is given in Euclid's own words /I, 402/; and the "proof" (a construction of the positive root of the equation, followed by a verification) is almost step by step the same as that given by Euclid. Book II, Proposition 11, is as follows:

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment. [Find H so that $a(a - x) = x^2$; in other words, find the positive root x (or AH) of the quadratic equation $x^2 + ax - a^2 = 0$.]

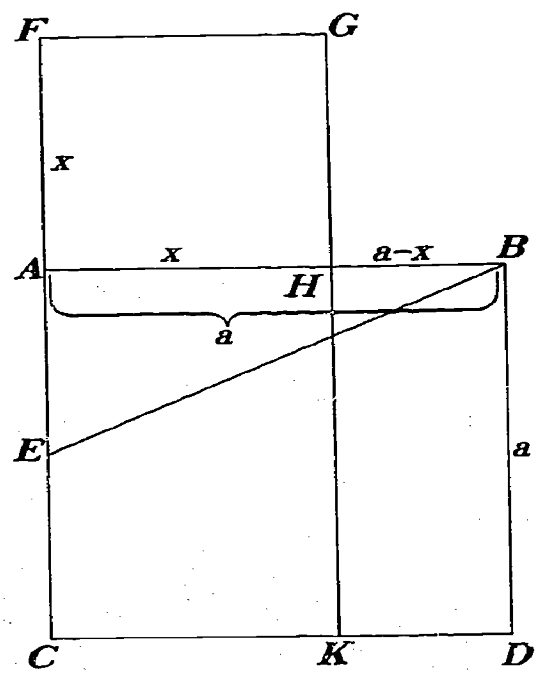

FIGURE [78]-2

AB, or a, is the given segment (Fig. [78]-2). Construct square $ABDC$. Bisect AC at E. Draw EB. Extend CA to F so that $EF = EB$. Construct square $FGHA$. Then H is the required point (so that $x = AH$ is the positive root of $x^2 + ax - a^2 = 0$).

Verification follows, modern notation being used in the right-hand column.

By an earlier proposition (II, 6)

$$CF \cdot FG + AE^2 = EF^2.$$

Prop. II, 6 is a form of the identity

$$(\alpha + \beta)(\alpha - \beta) + \beta^2 \equiv \alpha^2 \qquad (1)$$

or

$$(\alpha + \beta)(\alpha - \beta) \equiv \alpha^2 - \beta^2,$$

where, in the present context

By construction $EF = EB$; hence

$$CF \cdot FG + AE^2 = EB^2.$$

By the Pythagorean theorem,

$$\begin{array}{rcl} CF \cdot FG + AE^2 & = & AB^2 + AE^2 \\ - AE^2 & = & - AE^2 \\ \hline CF \cdot FG & = & AB^2 \\ - AHKC & = & - AHKC \\ \hline AH^2 & = & DB \cdot HB \end{array}$$

or

$$AH^2 = AB \cdot HB. \qquad (2)$$

$$\alpha = x + \frac{a}{2} \quad \text{and} \quad \beta = \frac{a}{2}$$

so that

$$\alpha + \beta = a + x \quad \text{and} \quad \alpha - \beta = x.$$

Hence (1) gives

$$(a + x)(x) + \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2.$$

By the Pythagorean theorem,

$$\begin{array}{rcl} (a + x)(x) + \left(\frac{a}{2}\right)^2 & = & a^2 + \left(\frac{a}{2}\right)^2 \\ - \left(\frac{a}{2}\right)^2 & = & - \left(\frac{a}{2}\right)^2 \\ \hline (a + x)(x) & = & a^2 \\ - ax & = & - ax \\ \hline x^2 & = & a(a - x). \end{array}$$

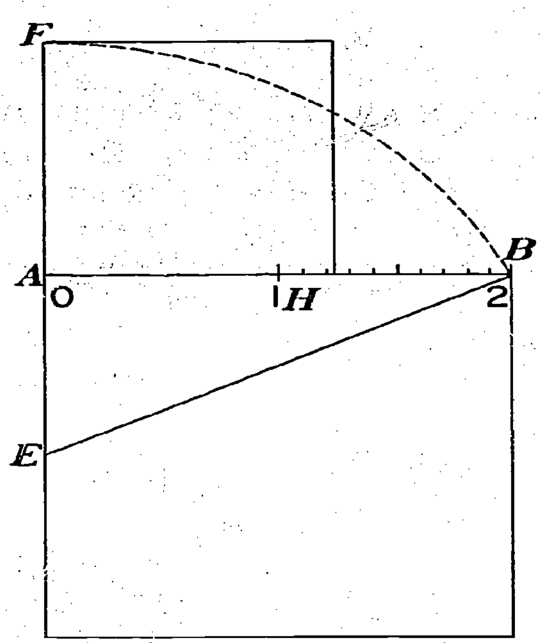

FIGURE [78]-3

Hence H is the required point (so that AH, or x, satisfies the condition (2)).

As an example, see Figure [78]-3. Let $AB = a = 2$ to get the quadratic equation $x^2 + 2x - 4 = 0$. Carrying through the above construction we find that $AH = x \approx 1.236$, which agrees with the positive root obtained from the quadratic formula, $x =$ [illegible] $\sqrt{5}$.

For Further Reading

AABOE (b): 61–65
EUCLID
EVANS
EVES (c): 64–69
[3d ed. 61–67]
VAN DER WAERDEN: 118–26

Capsule 79 *Ferna E. Wrestler*

HINDU ALGEBRA

THE Hindu work on astronomy *Surya Siddhanta* ("Knowledge from the Sun"), written around A.D. 500, provided the motivation for a remarkable development of arithmetic and algebra in India as shown by the works of Aryabhata (*c.* 525), Brahmagupta (628), Mahavira (*c.* 850), and Bhaskara (1150). After Bhaskara, Hindu mathematics showed no progress until modern times.

Brahmagupta gave an interesting rule for finding one of the two positive roots of the quadratic equation $x^2 - 10x = -9$ (using modern notation), which in the original is written as shown here:

$$ya\ v\ 1\ ya\ \dot{1}\dot{0}$$
$$ru\ \dot{9}$$

In this, *ya* is the unknown; *v* means "squared"; the dot above a number indicates that [illegible] a negative number. The left-hand member of the equation (as we would describe it) is written on one line and the

right-hand member beneath; *ru* means "absolute" ("plain") number.
The three columns below give the solution as translated /D. E. SMITH (a): II, 445/, then in modern notation and with a generalization for $ax^2 + bx = c$.

	$x^2 - 10x = -9.$	$ax^2 + bx = c.$
Here absolute number (9) multiplied by (1) the [coefficient of the] square [is] (9)	$(-9)(1) = -9$	$(c)(a) = ca$
and added to the square of half the [coefficient of the] middle term, 25, makes 16;	$-9 + \left(\frac{-10}{2}\right)^2 = 16$	$ca + \left(\frac{b}{2}\right)^2$
of which the square root 4, less half the [coefficient of the] unknown (5), is 9;	$\sqrt{16} - \left(\frac{-10}{2}\right) = 9$	$\sqrt{ca + \left(\frac{b}{2}\right)^2} - \frac{b}{2}$
and divided by the [coefficient of the] square (1) yields the value of the unknown 9.	$\frac{9}{1} = 9.$	$\frac{\sqrt{ca + \left(\frac{b}{2}\right)^2} - \frac{b}{2}}{a} = x$
		or
		$x = \frac{-b + \sqrt{b^2 + 4ac}}{2a}.$

The method used in the above example is essentially the same as our present method of "completing the square" and consists of adding the shaded area $(b/2)^2$ of Figure [79]-1 to the unshaded area

$$(a^2x^2 + abx) + \left(\frac{b}{2}\right)^2,$$

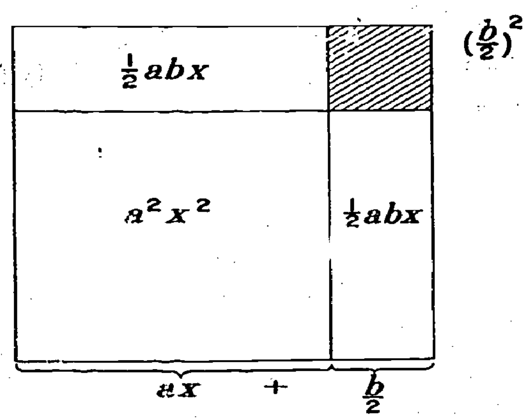

FIGURE [79]-1

which gives the whole area:

$$\underbrace{(a^2x^2 + abx)}_{ca} + \left(\frac{b}{2}\right)^2.$$

Since it was given that $ax^2 + bx = c$, put ca for $a^2x^2 + abx$; add $(b/2)^2$ to equal

$$\left(ax + \frac{b}{2}\right)^2.$$

Hence the side, $ax + (b/2)$, of the large (completed) square is

$$\sqrt{ca + \left(\frac{b}{2}\right)^2};$$

and ax is $b/2$ less than

$$\sqrt{ca + \left(\frac{b}{2}\right)^2}.$$

Finally, divide by a to obtain x.

The example given shows that Hindu algebra was largely verbal (rhetorical), although in the statement of the problem use is made of abbreviations, illustrating the so-called syncopated style. Especially noteworthy is the correct use of negative numbers, written by placing a dot above the number. Imaginary numbers escaped the Hindus, who, however, at least recognized them as rating a comment: "as in the nature of things, a negative is not a square, it has therefore no square root" (Mahavira). They operated freely with irrational numbers and used the identity that would be written in modern notation as

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

They realized that a quadratic equation with real roots would have two roots, but they did not always bother to find both roots, as we have seen. Negative roots were discarded as "inadequate."

The Hindus worked with arithmetic and geometric progressions, permutations, and linear equations; and they could solve some equations of degree higher than two.

The Hindus made their greatest progress in indeterminate analysis. For an equation $ax + by = c$ (a, b, and c integers) with an integral

solution, they could determine the solution by continued fractions, a method that is still used. After finding one solution, $x = p$, $y = q$, they found others by using $x = p + bt$, $y = q - at$ for any integer t. Likewise, if one pair of integers p and q could be found to satisfy a so-called Pell equation, $y^2 = ax^2 + 1$ (a an integer that is not a square), they could find more by using the following property /CAJORI (e): 95/: "If p and q is one set of values of x and y, and p' and q' is the same or a different set, then $qp' + pq'$ and $app' + qq'$ is another solution." A problem from Bhaskara's works is this: "What square number multiplied by 8 and having 1 added shall be a square?" One solution of the equation $8x^2 + 1 = y^2$ is $x = 6$, $y = 17$, from which it is readily seen that $x = 204$, $y = 577$ is another.

Of interest is Bhaskara's solution of a problem on right triangles. The problem is given as follows /SCOTT: 73/: "The hypotenuse being 85, say, learned men, what upright sides will be rational?" (In modern symbolism, "Find rational values of x and y if $x^2 + y^2 = h^2$.") The solution is given below, with modern symbolism at the right.

Double the hypotenuse.	170	$2h$
Multiply by an arbitrary number, say 2.	340	$2ah$
Divide by the square of the arbitrary number increased by 1.	$\frac{340}{5}$	$\frac{2ah}{a^2+1}$
This gives one side.	68	
Multiply by the arbitrary number, 2.	136	$\frac{2a^2h}{a^2+1}$
Subtract the hypotenuse.	$136 - 85$	$\frac{2a^2h}{a^2+1} - h$
This gives the other side.	51	$\frac{h(a^2-1)}{a^2+1}$

This is equivalent to saying that the three sides of a right triangle are proportional to $a^2 + 1$, $2a$, $a^2 - 1$; and the values are not unique but depend on the choice of the arbitrary number a.

The following problem is typical of this period: "The horses belonging to four persons are 5, 3, 6, 8, respectively. The camels pertaining to the same are 2, 7, 4, 1. The mules belonging to them are 8, 2, 1, 3, and the oxen 7, 1, 2, 1. All four persons being equally rich, tell

me the price of each horse and the rest." (One solution is horses, 85; camels, 76; mules, 31; oxen, 4.)

For Further Reading

CAJORI (e): 83–98
EVES (c): 181–91
[3d ed. 180–89]
MIDONICK: 116–40, 166–80
SCOTT: 66–80
D. E. SMITH (a): I, 152–64, 274–81

Capsule 80 *Cecil B. Read*

ARABIC ALGEBRA, 820-1250

LITTLE is known about Arabian history before the time of Mohammed (570–632). Mohammed was instrumental in forming a powerful nation that eventually extended over parts of India, Persia, Africa, and Spain. Baghdad was the Eastern intellectual center, and Cordova, in Spain, the Western.

The rulers, called caliphs, supported scientific research. The Arabs, conquering Egypt, acquired some Greek masterpieces from the Alexandrian library. Conquering part of India, they came in contact with the Hindus. Works of Hindu mathematicians were translated, and Hindu numerals entered Arabia. Greek mathematical works, including Euclid's *Elements* and the writings of Archimedes, Heron, Ptolemy, Apollonius, and Diophantus, were also translated into Arabic. Often Arabic translations of Hindu and Greek works are the only known copies.

Arabic algebra came from both the Hindus and the Greeks. The Arabs treated algebra numerically like the Hindus and geometrically like the Greeks.

The early Arabs wrote out problems entirely in words. After contact with other peoples, symbols and Hindu numerals were gradually introduced; but later Arabian writers reverted to writing out problems completely, showing perhaps the influence of Greek methods.

Possibly the greatest of the Arabic mathematical writers was al-Khowarizmi (c. 825), although some think his algebra shows little originality. He used a type of "transposition" that is not found in Hindu or Greek works, and he seems to have been the first to collect like powers of the unknown. These may be his original ideas. He solved linear and quadratic equations, both numerically and geometrically. He recognized the existence of negative roots (as the Hindus also did) but consciously rejected them.

The original Arabic edition of al-Khowarizmi's best-known work, *Hisab al-jabr w'al muqabalah,* is lost, but a Latin translation exists (dating from the twelfth century). One translation of the title is "The Science of Transposition and Cancellation." The book became known as *Al-jabr,* from which we get our word "algebra." Subsequent Arabic and medieval algebras were based on al-Khowarizmi's work.

The following example shows, in al-Khowarizmi's own words (as translated /D. E. SMITH (a): II, 447/), how he found the positive root of the quadratic equation that we would write as $x^2 + 10x = 39$. The second column shows this in numerical values, and the third gives a generalization for $x^2 + px = q$.

	$x^2 + 10x = 39.$	$x^2 + px = y.$
You halve the number of roots, which in the present instance yields five.	$\frac{1}{2}(10) = 5$	$\frac{p}{2}$
This you multiply by itself; the product is twenty-five.	$5 \cdot 5 = 25$	$\left(\frac{p}{2}\right)^2$
Add this to thirty-nine; the sum is sixty-four.	$25 + 39 = 64$	$\left(\frac{p}{2}\right)^2 + q$
Now take the root of this, which is eight,	$\sqrt{64} = 8$	$\sqrt{\left(\frac{p}{2}\right)^2 + q}$
and subtract from it half the number of the roots, which is five; the remainder is three.	$8 - \frac{10}{2} = 3$	$\sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2} = x$
This is the root of the square which you sought for; the square itself is nine.		$\left[\begin{array}{c}\text{or} \\ x = \frac{-p + \sqrt{p^2 + 4q}}{2}.\end{array}\right]$

The method used is essentially the same as our present-day method of "completing the square" and consists literally of adding the shaded

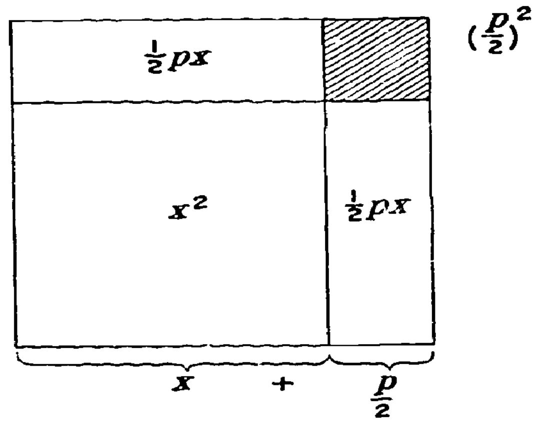

FIGURE [80]-1

area $(p/2)^2$ of Figure [80]-1 to the unshaded area $(x^2 + px)$ to complete the square of side $(x + (p/2))$. That is,

$$\left(\frac{p}{2}\right)^2 + (x^2 + px) = \left(x + \frac{p}{2}\right)^2;$$

but since it was given that $(x^2 + px) = q$, $(p/2)^2 + q = (x + p/2)^2$. Hence the side $(x + (p/2))$ of the completed square is equal to

$$\sqrt{\left(\frac{p}{2}\right)^2 + q};$$

and x is $p/2$ less than that quantity.

Notice that the *other* root, -13, of the equation $x^2 + 10x = 39$ was ignored because it is negative. If both roots had been positive they would probably have both been found.

Abu Kamil (*c.* 900) wrote a more extensive treatise on algebra. It was so good that later writers used much of it, although without mentioning his name. The methods were well known and considered to be common property. He used both the terms "square" and "root." The Greeks thought of 5 as the side of a square with area 25; the Arabs, following the Hindus, thought of 25 as growing, like a tree, out of the number 5 as a root. Both concepts appear in "square root." The Latin word for "root" is *radix*; from it comes our word "radical."

Like others, Abu Kamil solved equations algebraically and geometrically. He classified quadratic equations into six types, presenting no general method. To give a single example indicating that he did work of more than elementary difficulty: he showed, without using the modern notation employed here, the equality

$$\sqrt{12 \pm 2\sqrt{20}} = 10 \pm \sqrt{2}.$$

One of the best Arabic algebraists was Omar Khayyam (*c.* 1100), known usually only as the author of the *Rubaiyat*. He used geometric algebra, solving cubic equations by finding the intersections of conics. Some think that this was the greatest achievement in Arabic algebra. Omar Khayyam thought the cubic was insolvable by purely algebraic means.

His method of solving $x^2 + 10x = 39$ (as we write it) was really the same as al-Khowarizmi's, but we state it because of its historical interest: "Multiply half of the root by itself; add the product to the number and from the square root of this sum subtract half the root. The remainder is the root [side] of the square." /D. E. SMITH (a): II, 447./ It is not a coincidence that the same numbers (10 and 39) appear in the two examples. This particular problem was a favorite in the Arab schools of that time.

Note especially that Arab mathematicians would not have thought of the above example in our customary form, $x^2 + 10x - 39 = 0$, because they simply did not grasp negative numbers; this difficulty with negative numbers and the subtleties of zero products probably explains why solution by factoring came rather late (in the time of Thomas Harriot, 1631).

Some work was done with indeterminate equations by al-Karkhi (*c.* 1020), who tended to follow the style of the Greek mathematician Diophantus. As one problem he proposes this: "Find rational numbers x, y, and z such that $x^3 + y^3 = z^2$."

Arabic algebra used the rules of false position and of double false position [90]. They explained the rule of three, which today we call proportion. The Hindu mathematicians had used the terms, and the Arabs translated directly.

Some historians think the Arabs added little that was new, but all agree that throughout the Dark Ages the Arabs preserved the Greek and Hindu works for posterity. Without their translations, most of this prior work would be lost.

It was principally through the Arabs that algebra entered Europe. Hindu influence dominated, hence algebra came to Europe with little axiomatic foundation. Perhaps this explains why, until quite recently, geometry was based on postulates and theorems while elementary algebra emphasized method rather than logical foundations.

For Further Reading

BOYER (g): 249–69
CAJORI (e): 99–112
CARNAHAN (a)
COOLIDGE (c): 19–29
EVES (c): 191–95
[3d ed. 190–93]
——— (e)
KHOWARIZMI
MIDONICK: 418–34
OMAR KHAYYAM
D. E. SMITH (a): II, 446–48
STRUIK (e): 55–60

Capsule 81 Sister M. Stephanie Sloyen

ALGEBRA IN EUROPE, 1200-1850

EUROPEAN algebra was based directly upon Arabic algebra and developed rather slowly from what might be termed its beginning, around 1200, until the nineteenth century, when discoveries followed closely upon one another.

Much of the early work was done in Italy. There Fibonacci (Leonardo of Pisa) did a great deal to popularize Hindu-Arabic numerals with his book on arithmetic and algebra, *Liber abaci* ("Book of Calculation"), written in 1202. This book also contains the famous Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, . . . [22].

For the next few centuries there was very little algebraic activity in Europe; however, during the period from 1515 to 1545 it was again Italy that produced the algebraists. During that time many mathematics books were published in Italy, although mathematicians did not send their disoveries to journals for publication. They preferred to use their new knowledge in order to shine in public contests, challenging one another in problem solving. Scipione del Ferro, a professor at the University of Bologna, in 1515 devised a method of solving the cubic equation $x^3 + bx = c$, but he did not circulate his work. Niccolo Tartaglia solved the cubic equation $x^3 + ax^2 = c$ and then also the cubic $x^3 + bx = c$ (about 1535) and used his information in order to vanquish challengers [71]. Girolamo Cardano, a physician and mathematician who was called the "gambling scholar" by Oystein Ore /(a)/, obtained the solution from Tartaglia and made many improvements on Tartaglia's solution, solving (at least for positive roots)

all possible cases except the "irreducible" one. He then published the complete solution of all varieties of the cubic equation (except the irreducible case involving "imaginaries") in his *Ars magna*, giving full credit to Tartaglia. It was Ludovico Ferrari who successfully solved the general quartic equation. (Rafael Bombelli, a sixteenth-century Bolognese mathematician, made progress on the irreducible case of the cubic by recognizing in 1572 that *apparently* imaginary expressions like

$$\sqrt[3]{81 + 30\sqrt{-3}} + \sqrt[3]{81 - 30\sqrt{-3}}$$

were real—in this case, -6.)

Algebraists of the seventeenth century include Thomas Harriot, an Englishman who introduced the signs $<$ and $>$ /C. SMITH; EVES (a)/ and the use of aa for what we call a^2 and aaa for a^3. While we may think this awkward, it is an improvement over the *A cubum* of François Viète or even the *res cubum* of earlier times. William Oughtred, another Englishman, was responsible for the slide rule, the multiplication sign $\times$, and the sign :: for proportion.

René Descartes, a Frenchman, was one of the greatest mathematicians of this century and a prolific writer. His outstanding contribution was, of course, his work on plane analytic geometry, but he also improved the symbolism of algebra and introduced our present system of positive, integral exponents. A large part of Descartes's *La géométrie* consists of what we now call "theory of equations," and it contains Descartes's rule of signs for determining the number of positive and negative roots of an equation. Descartes used the letters at the end of the alphabet, . . . , x, y, z, for variables, and the early letters, a, b, c, . . ., for constants. Viète, in the sixteenth century, had used vowels for variables and consonants for constants.

Pierre de Fermat's work in the seventeenth century in France was chiefly in number theory; theorems in Diophantine analysis (of which he left no proof) are due to him. Isaac Newton, genius in many fields and inventor of the calculus, discovered the binomial theorem in 1664 when he was twenty-two. The theory of symmetric functions of the roots of an equation, first perceived by Viète, was firmly established by Newton, who also gave a method for finding approximations to the roots of numerical equations.

In the nineteenth century mathematicians began to work in specialized fields, but Carl Friedrich Gauss was an exception to this rule. In his doctoral dissertation, written when he was twenty and published

in 1799, he gave the first rigorous proof of the fundamental theorem of algebra: *Every algebraic equation of degree n has a root* [and hence n roots]. Later he published three more proofs of the same theorem. It was he who called it "fundamental." Much of the work on complex number theory is Gauss's. He was one of the first to represent complex numbers as points in a plane. From 1807 until his death in 1855, Gauss was director and professor of astronomy at the observatory in Göttingen, Germany, where he had graduated from the university.

Évariste Galois, killed in a duel in 1832 at the age of twenty-one, was a genius never recognized in France during his lifetime. On the eve of the duel he wrote to a friend /D. E. SMITH (c): 285/:

> Ask Jacobi or Gauss publicly to give their opinion, not as to the truth, but as to the importance of the theorems [see below]. Subsequently there will be, I hope, some people who will find it to their profit to decipher all this mess.

This note was attached to what Galois thought were some new theorems in the theory of equations; these turned out to contain the essence of the theory of groups, so important today. At about the same time Niels Henrik Abel, in Norway, thought he had found a method of solving the general quintic equation, but later he corrected himself and proved that a solution by means of radicals was impossible.

Finally, we take note of two English algebraists, Arthur Cayley and James Joseph Sylvester. As a young man Cayley practiced law in London, and it was there that he met Sylvester, an actuary. For the rest of their lives they worked together on the theory of algebraic invariants.

Although he spent most of his life in England, Sylvester brought his work to America (he taught briefly at the University of Virginia in 1841/42 and returned to the States to teach at Johns Hopkins University from 1877 to 1883). He established graduate study in mathematics in this country, and "American algebra" might be said to begin with him.

For Further Reading

BOYER (g): 333–38, 367–81, 544–49, 629–32
EVES (a)
ORE (a)
C. SMITH
STRUIK (e): 74–111, 115–22

FUNCTION

Definition 1.—A function is a set of ordered pairs whose first elements are all different.

Definition 2.—When the value of one variable depends on another, the first is said to be a function of the second.

Definition 3.—If to each permissible value of x there corresponds one or more values of y, then y is a function of x.

Definition 4.—If y is a function of x, then it is equal to an algebraic expression in x.

TWENTY elementary algebra texts were examined for definitions of "function"; eleven of these texts were published before 1959, nine after 1959. The older texts used Definitions 2, 3, 4, and others; six of the newer ones used Definition 1.

Fifteen college algebra texts were examined, seven published before 1959 and eight after 1959. None of the older texts used Definition 1; four of the eight newer ones did.

This quite recent history of "function" has additional significance in the context of the earlier history of both the idea and the word.

Eric Temple Bell suggests /(a): 32/ that the Babylonians of c. 2000 B.C. might be credited with a working definition of "function" because of their use of tables like the one for $n^3 + n^2$, $n = 1, 2, \ldots, 30$, suggesting the definition that a function is a table or correspondence (between n in the left column and $n^3 + n^2$ in the right column).

More explicit ideas of function seem to have begun about the time of René Descartes (1637), who may have been the first to use the term; he defined a function to mean any positive integral power of x, such as $x^2, x^3, \ldots$.

Gottfried Wilhelm von Leibniz (1692) thought of a function as any quantity associated with a curve, such as the coordinates of a point on a curve, the length of a tangent to the curve, and so on.

Johann Bernoulli (1718) defined a function to be any expression involving one variable and any constants.

Leonhard Euler (1750) called functions in the sense of Bernoulli's

definition "analytic functions" and used also a second definition, according to which a function was not required to have an analytic expression but could be represented by a curve, for example. Euler also introduced the now standard notation $f(x)$.

Joseph Louis Lagrange (1800) restricted the meaning of function to a power series representation. Jean Joseph Fourier (1822) stated that an arbitrary function can be represented by a trigonometric series. P. G. Lejeune Dirichlet (1829) said that y is a function of x if y possesses one or more definite values for each of certain values that x may take in a given interval, x_0 to x_1.

More recently, the study of point sets by Georg Cantor and others has led to a definition of function in terms of ordered pairs of elements, not necessarily numbers.

For Further Reading

BELL (a): *See index*
BOYER (f): 243, 276–77
——— (g): 290–92
CAJORI (d): II, 267–70
EVES (c): 371–72
[3d ed. 371–72]
Growth of Mathematical Ideas 65–110, 445–49
Insights into Modern Mathematics: 55–58, 220, 241–72, 409–11
F. KLEIN (a): I, 200–207
G. A. MILLER (b)
READ (e)
Selected Topics: 42–56
JOHN YOUNG: 192–200

Capsule 83

MATHEMATICAL INDUCTION

FROM the mathematical experiment

$$1 + 3 = 2^2,$$

$$1 + 3 + 5 = 3^2,$$

$$1 + 3 + 5 + 7 = 4^2,$$

etc.,

one is led to the formula

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2;$$

and then this conjecture is proved deductively by using the principle of mathematical induction.

B. L. Van der Waerden /126/ points out that "in essence" the principle of mathematical induction was known to the Pythagoreans but that Francesco Maurolico was the first to make fairly explicit use of it (in his *Arithmetic*, 1575). Blaise Pascal (*c.* 1653) was the next person to use the idea, as he did repeatedly in his work on the so-called Pascal triangle, which he called the "arithmetic triangle."

The induction proofs of Maurolico are given in a rather sketchy style not easily followed. Pascal's style is more nearly along modern lines, and we present in modern notation a translation of his induction proof that

$$\frac{{}_nC_r}{{}_nC_r + 1} = \frac{r + 1}{n - r},$$

where

$${}_nC_r = \frac{n!}{(n - r)!\,r!}$$

and r is any "cell" from the 0th to the $(n - 1)$th in Figure [83]-1.

Consequence XII: *In every arithmetic triangle, two adjoining cells on the same line* [have the property that] *the lower is to the higher as the number of cells below (and including) the lower is to the number of cells above (and including) the higher.*

Let E and C be any two adjoining cells on the same line; I say that

E	is to	C	as	2	to	3
lower		higher		because there are two cells from E to the bottom, namely, E, H.		because there are three cells from C to the top, namely, C, R, μ.

Although this proposition has an infinity of cases, I shall give a very short demonstration based on two lemmas:

The first, which is self-evident, that this proportion is true on the second line [of the triangle]; because it is easily seen that ϕ is to σ as 1 is to 1 [let $n = 1$; then ${}_1C_0/{}_1C_1 = (0 + 1)/(1 - 0)$].

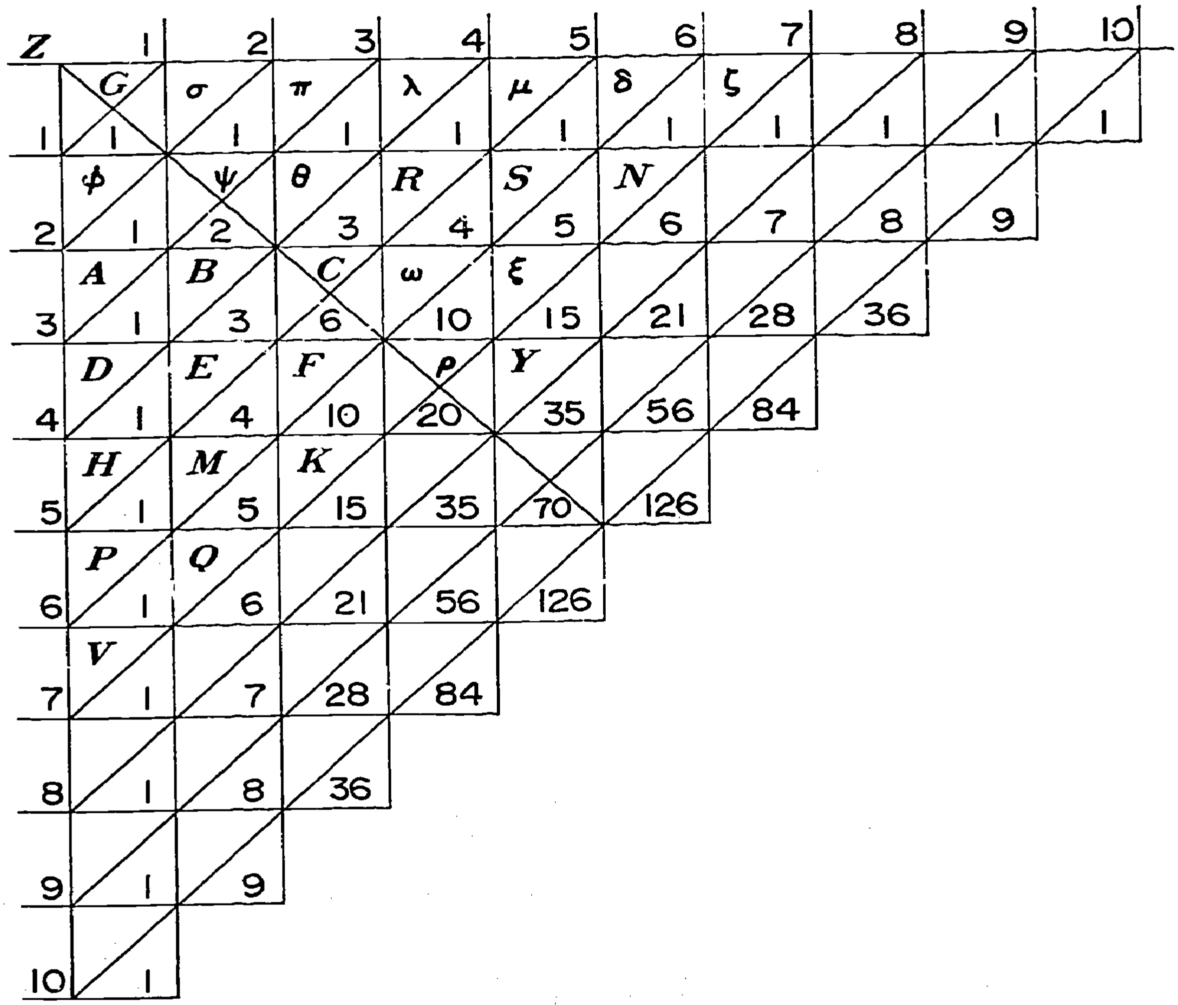

FIGURE [83]-1

The second, that if this proportion is true on any line it will necessarily be true on the following line. [Let $n = k$. Then

$${}_kC_r/{}_kC_{r+1} = (r+1)/(k-r)$$

implies

$${}_{k+1}C_r/{}_{k+1}C_{r+1} = (r+1)/((k+1)-r),$$

and hence the theorem is true for $n = k + 1$ *if* it is true for $n = k$.] From which it is apparent that it is necessarily true on all the lines: for it is true on the second line by the first lemma; therefore by the second [lemma] it is true on the third line; therefore on the fourth, and so on.

It is necessary therefore only to prove the second lemma in this way: If the proportion is true on any line, as on the fourth D λ; for example, if D is to B as 1 to 3, and B to θ as 2 to 2, and θ to λ as 3 to 1,

and so forth, I say that this same proportion will be true on the following line $H\mu$ and that, for example, E is to C as 2 to 3.

For D is to B as 1 to 3 by hypothesis.

Therefore $\underbrace{D + B}_{E}$ is to B as $\underbrace{1 + 3}_{4}$ to 3

E to B as 4 to 3.

Likewise B is to θ as 2 to 2 by hypothesis.

Therefore $\underbrace{B + \theta}_{C}$ is to B as $\underbrace{2 + 2}_{4}$ to 2

C to B 4 to 2.

But E to B as 4 to 3 [and B to C as 2 to 4] (as was shown). Then [by multiplying these last two proportions] E is to C as 2 to 3. Which it was required to show.

One can show the same on all the rest [of the lines], since this proof is based only on that proportion found for the preceding [line], and [the property] that each cell is equal to its preceding [one on the left] plus the one above it, which is true everywhere [in the triangle].

The "property" referred to is, for example,

$$E = D + B;$$

or, in general,

$${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r,$$

which is Pascal's rule of formation (definition) of the arithmetic triangle.

For Further Reading

MESCHKOWSKI (b): 36–43

D. E. SMITH (c): I, 67–79

STRUIK (e): 21–26

Capsule 84

FUNDAMENTAL THEOREM OF ALGEBRA

CARL Friedrich Gauss, at the age of twenty (1797), gave the first satisfactory proof of the theorem which he called fundamental and which was the topic for his doctoral dissertation at the University of

Helmstädt, *A New Proof that Every Rational Integral Function of One Variable Can Be Resolved into Real Factors of the First or Second Degree.* (Equivalent statements are "Every algebraic equation of degree n has n roots," and "Every algebraic equation of degree n has a root of the form $a + bi$, where a and b are real.") Actually, Gauss gave four proofs for the theorem, the last when he was seventy; in the first three proofs he assumes the coefficients of the polynomial equation are real, but in the fourth proof the coefficients are any complex numbers.

The words "new proof" in Gauss's title indicate that the ideas summarized in the statement of the theorem had been considered by earlier mathematicians. The Hindus (by 1100 at the latest) realized that quadratic equations (with real roots) had two roots. Girolamo Cardano realized in 1545, though somewhat vaguely because negative and imaginary numbers were not clearly defined at this time, that cubics should have three roots; and he exhibited three roots for some cubics. Similar ideas were held with respect to quartic equations by Cardano and other Italian algebraists of this period.

François Viète (*c.* 1600) considered the possibility of factoring the left member of the polynomial equation $f(x) = 0$ (with positive coefficients) into linear factors but was foredoomed to only partial success because of his marked aversion to negative and imaginary numbers.

Peter Roth seems to have been the first writer to say definitely that a polynomial equation of degree n has n roots. This was in 1608. Albert Girard stated in 1629 that every algebraic equation has as many roots as the degree of its highest power.

The insights of René Descartes on this matter are of special interest because they are related to his famous "rule of signs." We quote from his *La géométrie* (1637) /(b): 159–60/:

> Every equation can have as many distinct roots (values of the unknown quantity) as the number of dimensions [i.e., degree] of the unknown quantity in the equation. . . .
>
> It often happens, however, that some of the roots are false or less than nothing. . . .
>
> We can determine also the number of true [positive] and false [negative] roots that any equation can have, as follows: An equation can have as many true roots as it contains changes of sign . . . and as many false roots as the number of times two + or two − signs are found in succession.

The first attempt at a proof seems to have been made by Jean Le Rond d'Alembert, in 1746, and for this reason the theorem is sometimes called d'Alembert's theorem, especially in France. Leonhard Euler (1749) and Joseph Louis Lagrange also tried to prove the theorem.

A correct proof was not given until Gauss wrote his doctoral dissertation, which was published in 1799. This included "geometrically obvious" assumptions for which later standards of rigor required proof, which A. Ostrowski gave in 1920.

For Further Reading

BELL (a): 178
——— (d): 218–69
COURANT and ROBBINS: 269–71
DUNNINGTON
F. KLEIN (a): I, 101–4
D. E. SMITH (c): I, 292–306
STRUIK (e): 81–87, 99–102, 115–22

Capsule 85 Donald W. Western

DESCARTES'S RULE OF SIGNS

IN 1637 the French philosopher René Descartes (1596–1650) published a book with a lengthy title commonly abbreviated to *Discours de la méthode,* a full translation being "Discourse on the Method of Rightly Conducting One's Reason and Seeking Truth in the Sciences." Three appendixes were included: *La dioptrique* ("Optics"), *Les météores* ("Meteorology"), and *La géométrie* ("Geometry"). The third part of the third appendix is entitled, in translation, "On the Construction of Solid and Supersolid Problems." It deals with many basic ideas for solving equations that arise in connection with geometric problems (primarily the study of conic sections by algebraic methods).

After posing some problems on mean proportions, Descartes proceeds to construct a fourth-degree polynomial equation by multiplying together the linear factors $(x - 2)$, $(x - 3)$, $(x - 4)$, and $(x + 5)$ to obtain

$$x^4 - 4x^3 - 19x^2 + 106x - 120 = 0.$$

He remarks that the polynomial is divisible by no other binomial factors and that the equation has "only the four roots 2, 3, 4, and 5." The fact that the fourth root is −5 rather than 5 is recognized by speaking of 5 as a "false" root, in contrast to the positive numbers, which are called "true" roots. (The minus sign is not used by Descartes to designate negative numbers.) Then comes the statement of the celebrated rule of signs /DESCARTES (b): 160/:

> We can determine also the number of true and false roots that any equation can have, as follows: An equation can have as many true roots as it contains changes of sign, from + to − or from − to +; and as many false roots as the number of times two + signs or two − signs are found in succession.

Following this general comment, Descartes points out the three changes of sign and the one succession (permanence) of sign in his example and concludes, "On connoit qu'il y a trois vraies racines; et une fausse"; that is "We know there are three true roots and one false root."

As is often the case with the promulgation of a significant mathematical result, this first statement of the relation between changes in signs of the successive terms of the polynomial and the nature of the roots was not complete. Neither was any attempt made at proof, other than the illustrative example that accompanied it.

There is some disagreement in the literature whether the rule of signs was generally known before Descartes's publication of *La géométrie*. Smith and Latham state in a footnote of their translation /DESCARTES (b): 160/ that [Thomas] Harriot had given it in his *Artis analyticae praxis*, published in London in 1631. However, Moritz Cantor denies this possibility, since Harriot did not admit negative roots. Girolamo Cardano (1501–1576) had stated a relation between one or two variations in sign and the occurrence of positive roots.

The process of refining the rule of signs continued over a period of two centuries. In this process two points, specifically, were clarified: (1) the fact that variations in sign determine only upper bounds for the number of positive roots because of the possibility of imaginary roots and (2) the fact that the permanences of sign determine bounds for the number of negative roots only for a complete polynomial—that is, one with no coefficients equal to zero.

Isaac Newton, in his work *Arithmetica universalis* (published in 1707 but written some thirty years earlier), gave an accurate statement

of the rule of signs and presented without proof a procedure for determining the number of imaginary roots. At about the same time Gottfried Wilhelm von Leibniz pointed out a line of proof, although he did not give it in detail. In 1675 Jean Prestet published an insufficient proof; Johann Andreas Segner published one proof in 1725 or 1728 and in 1756 a more complete one. In 1741 Jean Paul de Gua de Malves gave a demonstration, introducing the argument that is basic to modern proofs. (This type of argument was employed more clearly by Segner in 1756.) Several other proofs were given in the period from 1745 to 1828. In 1828 Carl Friedrich Gauss added the significant contribution to the statement of the rule that if the number of positive roots falls short of the number of variations, it does so by an even integer.

The complete statement of Descartes's rule of signs is as follows:

> *Let* $P_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$, *where the coefficients* $a_0, a_1, \cdots, a_n$ *are real numbers,* $a_0 \neq 0$. *Then the number of positive real roots of the equation* $P_n(x) = 0$ [a root of multiplicity m being counted m times] *is either equal to the number of variations in signs or less than that number by a positive even integer.*

T' gative roots of $P_n(x) = 0$ is handled simply by consic e , e roots of $P_n(-x) = 0$. Thus the matter of permanenc is avoided.

The crux of the proof stems from the work of Gua de Malves and Segner. It consists in showing that if

$$P_n(x) = (x - r)P_{n-1}(x),$$

where $P_{n-1}(x)$ has real coefficients and r is positive, then $P_n(x)$ has at least one more variation in sign than does $P_{n-1}(x)$—for the general case, an odd number more.

For Further Reading

Bell (d): 35–55	Descartes (b)
Cajori (e): 178–79, 248	Struik (e): 89–99

SYMMETRIC FUNCTIONS

A SYMMETRIC function of two or more variables is a function that is not affected if any two of the variables are interchanged. Perhaps the most familiar symmetric functions are those met in elementary theory of equations where for the cubic equation

$$x^3 + C_1x^2 + C_2x + C_3 = 0$$

we have

$$r_1 + r_2 + r_3 = -C_1, \qquad r_1r_2 + r_1r_3 + r_2r_3 = C_2, \qquad r_1r_2r_3 = -C_3.$$

These last three equalities express the coefficients of the cubic equation as symmetric functions of the roots r_1, r_2, r_3.

When François Viète made his first tentative discoveries concerning symmetric functions in the late sixteenth century, the very notion of the roots of an algebraic equation was incomplete, in large measure because of an incomplete understanding of negative and imaginary numbers. Viète himself worked only with positive roots. He noticed that if the equation $x^3 + b = ax$ $(a > 0, b > 0)$ has two *positive* roots, r_1 and r_2, then

(1) $$r_1^2 + r_2^2 + r_1r_2 = a,$$

and

(2) $$r_1r_2(r_1 + r_2) = b.$$

Also, as Cajori says /(b) : 230/:

> His nearest approach to complete recognition of the facts is contained in the statement that the equation
>
> $$x^3 - (u + v + w)x^2 + (uv + vw + wu)x - uvw = 0$$
>
> has three roots, u, v, w. For cubics, this statement is perfect, if u, v, w are allowed to represent any numbers. But Viète is in the habit of assigning to letters only positive values, so that the passage really means less than at first sight it appears to do.

Albert Girard was interested in extending Viète's result. He considered all roots—those he called "impossible" (i.e., imaginary) as well as negative and positive roots. He studied the sums of their products taken two at a time (analogous to Viète's (1), above), then three at a time (analogous to Viète's (2), above), and so on.

But Girard was also interested in obtaining expressions for the sums of given powers of the roots; these sums constituted a different set of symmetric functions than the one Viète had essentially pioneered. Girard published his results in Amsterdam in 1629 in a pamphlet *Invention nouvelle en l'algèbre,* which contained the statement /FUNKHOUSER: 361/ that if

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \cdots = 0,$$

then

$$\left.\begin{array}{l} A \\ A^2 - 2B \\ A^3 - 3AB + 3C \\ A^4 - 4A^2B + 4AC + 2B^2 - 4D \end{array}\right\} \begin{array}{c} \text{will be} \\ \text{the sum of} \end{array} \left\{\begin{array}{l} \text{solutions} \\ \text{squares} \\ \text{cubes} \\ \text{biquadrates.} \end{array}\right.$$

Girard stated this result rather casually. Perhaps because of this and perhaps also because seventeenth-century mathematicians were not ready, Girard's remark went unnoticed until it reappeared, without proof, in Isaac Newton's *Arithmetica universalis* (1707) and became famous. It also became one of several theorems that are called "Newton's theorem."

For a hundred years after Newton many mathematicians, including Colin Maclaurin, Leonhard Euler, and Joseph Louis Lagrange, concerned themselves with proofs and generalizations of this theorem.

For Further Reading

CAJORI (b): 230–31
FUNKHOUSER
STRUIK (e): 81–87

DISCRIMINANT

As A result of the historical development of ideas leading to the term "discriminant," there is today a slight inconsistency in the use of the word. Texts dealing with the equation

$$Ax^2 + Bx + C = 0$$

call $B^2 - 4AC$ the discriminant of the equation. Other texts, discussing the binary quadratic form

$$Q(x, y) = Ax^2 + 2Bxy + Cy^2,$$

call $AC - B^2$ the discriminant of Q. Similar though these expressions are, the first is negative four times what we would expect it to be if notations were uniform. And even if this were corrected, it would not be immediately obvious that we are justified in using the same name—that we have the same mathematical entity.

By the middle of the eighteenth century it was well known that a necessary and sufficient condition for the equation $Ax^2 + Bx + C = 0$ to have two identical roots was $B^2 - 4AC = 0$. The expression was known; mathematicians knew what it signified and how to work with it; but it was not yet recognized as a mathematical entity.

During the next hundred years mathematicians studied several expressions related to the quadratic form. In 1748 Leonhard Euler used conditions involving expressions like those above to determine whether a quadric surface is contained in finite space; but Euler did not give a name to these expressions.

The expression that was not yet an entity reappeared in 1773. Joseph Louis Lagrange was studying the binary quadratic form given above. He proved that if $x + \lambda y$ were substituted for x, leading to a new form

$$A(x + \lambda y)^2 + 2B(x + \lambda y)y + Cy^2,$$

then if the new expression is simplified to

$$A'x^2 + 2B'xy + C'y^2,$$

we must have

$$A'C' - B'^2 = AC - B^2.$$

Other mathematicians turned to the study of such invariants, and similar expressions kept reappearing. Carl Friedrich Gauss called such an expression a "determinant" of the function. It remained for the tempestuous James Joseph Sylvester, who called himself the "mathematical Adam" because of his habit of giving names to mathematical creatures, to name this one. In 1851 he was studying invariants in reducing certain sixth-degree functions of two variables to simpler forms. What he found was what he called (and what we now recognize as) the "discriminant of a cubic."

His explanation in a long, testy, and somewhat defensive footnote is amusing and enlightening:

> "Discriminant," because it affords the *discrimen* or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally of the existence or otherwise of multiple points in the locus represented or characterized by any algebraical function, the most obvious and first observed species of singularity in such function or locus. Progress in these researches is impossible without the aid of clear expression; and the first condition of a good nomenclature is that different things should be called by different names. The innovations in mathematical language here and elsewhere (not without high sanction) introduced by the author, have been never adopted except under actual experience of the embarrassment arising from the want of them, and will require no vindication to those who have reached that point where the necessity of some such addition becomes felt.

Both our cases satisfy Sylvester's definition. The discriminant is a combination of constants which vanishes if at least two factors of a function are the same. If

$$B^2 - 4AC = 0, \qquad A \neq 0,$$

then

$$Ax^2 + Bx + C = A(x + B/2A)^2;$$

under the same conditions (or equivalently, with changed notation, if $AC - B^2 = 0, A \neq 0$),

$$Ax^2 + 2Bxy + Cy^2 = \frac{1}{A}(Ax + By)^2.$$

For Further Reading

BELL (d): 378–405 BOYER (g): 253, 258

Capsule 88 L. S. Shively

INTEREST AND ANNUITIES

IN THE *Liber abaci* of Fibonacci (Leonardo of Pisa), written in 1202, the following problem appears /EVES (c): 234/:

> A certain man puts one denarius at [compound] interest at such a rate that in five years he has two denarii, and in every five years thereafter the money doubles. I ask how many denarii he would gain from this one denarius in one hundred years?

The answer, $(2^{20} - 1)$ denarii, is easily obtained, since exactly 20 doublings are involved. The implied interest rate of 16⅓ percent compounded annually is possibly a commentary on the rather high rates charged in medieval Europe in spite of certain restrictions by the Church.

The custom of charging interest is found as early as 2000 B.C., as recorded on ancient Babylonian clay tablets. We give one example: /D. E. SMITH (a): II, 560/:

> Twenty manehs of silver, the price of wool, the property of Belshazzar, the son of the king. . . . All the property of Nadin-Merodach in town and country shall be the security of Belshazzar, the son of the king, until Belshazzar shall receive in full the money as well as the interest upon it.

Interest rates in Babylonia ran as high as 33 percent. In Rome during Cicero's day 48 percent was allowed; Justinian later set the maximum allowable rate at 0.5 percent per month, which gave rise to the common rate of 6 percent a year. In India, however, during the twelfth century, rates as high as 60 percent are recorded.

The origin of the word "interest" is related to church policy, which forbade usury, payment for the use of money. The moneylender got

around this restriction of canon law by collecting a fee only if the money was repaid tardily (which happened often enough even in those days!). The lender argued that the fee compensated him for the monetary difference between his poorer financial standing, because of late payment, and what would have been the standing under prompt repayment. This difference was referred to as *id quod interest* ("that which is between").

Annuities were known as early as 1556, the year in which Niccolo Tartaglia, in his *General trattato,* gives the following problem, which he said was brought to him by gentlemen from Barri who said that the transaction had actually taken place /SANFORD (d): 136/:

> A merchant gave a university 2,814 ducats on the understanding that he was to be paid 618 ducats a year for nine years, at the end of which the 2,814 ducats should be considered as paid. What interest was he getting on his money?

The answer to the problem is that the interest rate was slightly more than 19 percent; but without logarithms and annuity tables, it was not considered easy.

In 1693 Edmund Halley, who is best known for his work as an astronomer, contributed to the study of life insur[illegible]uities with the publication of [illegible]grees of Mortality of Mankind . . . with an Attempt to Ascertain the Price of Annuities upon Lives*. This included the following formula /CAJORI (e): 171/:

> To find the value of an annuity, multiply the chance that the individual concerned will be alive after n years by the present value of the annual payment due at the end of n years; then sum the results thus obtained for all values of n from 1 to the extreme possible age for the life of that individual.

Halley probably used the mortality table published in 1662 by John Graunt of London in his *Natural and Political Observations . . . Made upon the Bills of Mortality*, which was based on records of deaths that were kept in London beginning in 1592. (These records were originally intended to keep track of deaths due to the plague.)

For Further Reading

SANFORD (d): 127–31 D. E. SMITH (a): II, 559–65

Capsule 89 *Donald J. Dessart*

EXPONENTIAL NOTATION

THE great French mathematician René Descartes is credited with first introducing, in about 1637, the use of Hindu-Arabic numerals as exponents on a given base. To any modern schoolboy the idea of writing $x \cdot x \cdot x$ as x^3 or $x \cdot x \cdot x \cdot x$ as x^4 seems so obvious that it is quite natural for one to feel that Descartes probably hit upon this idea without help from his many predecessors in mathematics. But that was not the case! Ingenious inventions very often result from the insights of men who have learned from the trials and errors of others; such was the case with Descartes's use of exponents.

In this short capsule we shall look at some examples of early exponential symbolism and shall see that the idea of an exponent was available when Descartes took the very significant step of using Hindu-Arabic numerals placed to the upper right of the base.

Sometime around 1552 an Italian mathematician, Rafael Bombelli, worked very diligently on a manuscript that he published in 1572 as an algebra book called *L'Algebra*. In this volume he wrote the solution to a problem, beginning it as shown below:

$$\mathbf{4.p.R.q.}\ \lfloor \mathbf{24.m.}\overset{\overset{1}{\smile}}{\mathbf{20}}\mathbf{,}\rfloor\ \mathbf{Eguale\ à}\overset{\overset{1}{\smile}}{\mathbf{2}}\mathbf{.}$$

A first glance at this line of symbols might lead one to think that Bombelli was using a very complicated secret code, as in a sense he was; he was writing the equation we represent by writing

$$4 + \sqrt{24 - 20x} = 2x.$$

Let us pause for a moment and compare Bombelli's equation with our present-day form. It is easy to see that "Eguale à" probably means "equals." Continuing, we can see that "p." probably stands for "plus" and "m." for "minus." The symbol "R.q." represents "square root"; the two angular symbols mean the same as parentheses in modern symbolism. Thus "R.q.⌊ ⌋" means the square root of the polynomial

written within the symbols. To write positive integral powers of a variable, x, Bombelli wrote the exponent in a small circular arc above a numeral, so that

$$\overset{\overset{1}{\smile}}{2},\ \overset{\overset{2}{\smile}}{2},\ \overset{\overset{3}{\smile}}{2}$$

meant the same as $2x$, $2x^2$, and $2x^3$ in modern notation. Thus

$$\text{R.q.}\ \lfloor 24.\text{m}.\overset{\overset{1}{\smile}}{20}\rfloor$$

means $\sqrt{24 - 20x}$.

At first thought, it may seem that Bombelli's method is much better than our present symbolism because he did not need to write the letter x. But suppose a mathematician wished to represent $x^2 - y^2$. Could he do this by writing

$$\overset{\overset{2}{\smile}}{1}.\text{m}.\overset{\overset{2}{\smile}}{1}\ ?$$

No! For this reason, Bombelli's exponents were short-lived.

The complete solution as Bombelli included it in *L'Algebra* is given in the left-hand column below. Cover the modern version in the right-hand column if you want to test your skill in translating.

4. p. R. q. $\lfloor 24.\ \text{m}.\ \overset{\overset{1}{\smile}}{20},\rfloor$ Eguale à $\overset{\overset{1}{\smile}}{2}.$	$4 + \sqrt{24 - 20x} = 2x.$
R. q. $\lfloor 24.\ \text{m}.\ \overset{\overset{1}{\smile}}{20}\rfloor$ Eguale à $\overset{\overset{1}{\smile}}{2}.$ m. 4.	$\sqrt{24 - 20x} = 2x - 4.$
24. m. $\overset{\overset{1}{\smile}}{20}$ Eguale à $\overset{\overset{2}{\smile}}{4}.$ m. $\overset{\overset{1}{\smile}}{16}.$ p. 16.	$24 - 20x = 4x^2 - 16x + 16.$
24. p. $\overset{\overset{1}{\smile}}{16}$ Eguale à $\overset{\overset{2}{\smile}}{4}.$ p. $\overset{\overset{1}{\smile}}{20}.$ p. 16.	$24 + 16x = 4x^2 + 20x + 16.$
24 Eguale à $\overset{\overset{2}{\smile}}{4}.$ p. $\overset{\overset{1}{\smile}}{4}.$ p. 16.	$24 = 4x^2 + 4x + 16.$
8 Eguale à $\overset{\overset{2}{\smile}}{4}.$ p. $\overset{\overset{1}{\smile}}{4}.$	$8 = 4x^2 + 4x.$
2	

Eguale à $\overset{\underset{\smile}{2}}{1.}$ p. $\overset{\underset{\smile}{1}}{1.}$	$2 = x^2 + x.$
$2\frac{1}{4}$	
Eguale à $\overset{\underset{\smile}{2}}{1.}$ p. $\overset{\underset{\smile}{1}}{1.}$ p. $\frac{1}{4}$.	$2\frac{1}{4} = x^2 + x + \frac{1}{4}.$
$1\frac{1}{2}$	
Eguale à $\overset{\underset{\smile}{1}}{1.}$ p. $\frac{1}{2}$	$1\frac{1}{2} = x + \frac{1}{2}.$
1	
Eguale à $\overset{\underset{\smile}{1}}{1.}$	$1 = x.$

Bombelli was not the only mathematician before Descartes to write a numeral above the coefficient to indicate the power of the variable. Nicolas Chuquet, a physician in Lyons, France, wrote 12^0, 12^1, 12^2, and 12^3 to designate 12, $12x$, $12x^2$, and $12x^3$ in his *Le triparty en la science des nombres,* written about 1484. He also used

$$12.^{1.\bar{m}}$$

to designate $12x^{-1}$. Later, about 1610, Pietro Cataldi wrote

$$\not{0},\ \not{2},\ \not{3},\ \not{4},$$

to stand for x^0, x^2, x^3, and x^4; and in 1593 the Dutch writer Adrianus Romanus used

$$1(\overline{45})$$

for x^{45}. In 1619 the Swiss mathematician Jobst Bürgi used Roman numerals as exponents. He wrote

$$\overset{\text{vi}}{8} + \overset{\text{v}}{12} - \overset{\text{iv}}{9} + \overset{\text{iii}}{10}$$

to indicate the polynomial

$$8x^6 + 12x^5 - 9x^4 + 10x^3.$$

The accompanying table summarizes the main items in the historical development of exponents, including negative and fractional exponents. Cardano's verbal notation and J. Buteo's pictorial notation illustrate styles otherwise omitted because they did not contribute to the development of our present system.

The history of the development of exponential notation is a credit to man's genius in finding facile symbolisms for expressing mathematical concepts.

TABLE [89]-1

HISTORICAL DEVELOPMENT OF EXPONENTS

Modern Notation	$7x$	$7x^2$	$7x^3$	Commentary
1360 Oresme				Different manuscripts show different notations. For $2^{1/2}$ he wrote $\frac{1}{2}2^{p}$; also $\boxed{\frac{1.p}{2.2}}$. For $9^{1/3}$, he wrote $\frac{1}{3}9^{p}$. For $(2\frac{1}{2})^{1/4}$, he wrote $\boxed{\frac{1.p\ 1}{4.2.2}}$. For $4^{3/2}$, he wrote $\boxed{1^{p}\ \frac{1}{2}}$ 4, also $\boxed{\frac{p.1}{1.2}}$ 4.
1484 Chuquet	7^1	7^2	7^3	For 7, Chuquet wrote 7^0. For $12x^{-1}$, he wrote $12.^{1.\bar{m}}$.
1545 Cardano	7. pos.	7. quad.	7. cub.	For $7x^4$, Cardano wrote 7 quadr. quad.
1559 Buteo	7ρ	7 ◇	7 □	
1572 Bombelli	$\overset{\underset{\smile}{1}}{7}$	$\overset{\underset{\smile}{2}}{7}$	$\overset{\underset{\smile}{3}}{7}$	
1585 Stevin	7①	7②	7③	Stevin suggested for $x^{\frac{3}{2}}$ the notation $\frac{3}{2}$ when he said, "3/2 in a circle would be the symbol for the square root of ③ [i.e., x^3]"; but he never used this notation.
1590 Viète	7N	7Q	7C	Viète used vowels for unknowns and consonants for constants (except that N, Q, C had already been reserved for powers). He used the first style for polynomial equations in one unknown with numerical coefficients. Both styles are from later editions of his work; ear-
Also	B in A 7 for 7 BA	B in A q 7 for 7 BA²	B in A cu 7 for 7 BA³	

				lier he wrote "B in A quadratum 7" for "B in A q7" [$7BA^2$]. In the second style, Viète wrote "B in A qq 7" for $7BA^4$ and "B in A q cu 7" for $7BA^5$.
1610 Cataldi	$7\ \not{1}$	$7\ \not{2}$	$7\ \not{3}$	For 7, Cataldi wrote 7 $\not{0}$.
1619 Bürgi	$\overset{\text{i}}{7}$	$\overset{\text{ii}}{7}$	$\overset{\text{iii}}{7}$	For $7x^4$, Bürgi wrote $\overset{\text{iv}}{7}$.
1631 Harriot	$7a$	$7aa$	$7aaa$	
1634 Herigone	$7a$	$7a2$	$7a3$	
1637 Descartes	$7x$	$7xx$	$7x^3$	For $7x^4$, Descartes wrote $7x^4$.
1656 Wallis	$7a$	$7aa$	$7a^3$	For $7x^4$, Wallis wrote $7a^4$. In 1676 Wallis spoke of $\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}$, etc. as having index $-\frac{1}{2}$. However, he never used this notation.
1676 Newton*	$7x$	$7xx$	$7x^3$	For $7x^4$, Newton used $7x^4$. He also used the following notations: $a^{\frac{1}{2}}, a^{\frac{3}{2}}, a^{\frac{5}{3}}$ etc.; a^{-1}, a^{-2}, a^{-3}, etc.; $\overline{P+PQ}\rvert^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \cdots$ where $A = P^{\frac{m}{n}}$; $\overline{x^{\sqrt{2}} + x^{\sqrt{7}}}\rvert^{\sqrt[3]{\frac{2}{3}}}$.

* In a letter (June 13, 1676) to Henry Oldenburg, secretary of the Royal Society of London, Newton said: "Since algebraists write a^2, a^3, a^4, etc. for aa, aaa, $aaaa$, etc., so I write $a^{1/2}$, $a^{3/2}$, $a^{5/3}$, for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c\ a^5}$; and I write a^{-1}, a^{-2}, a^{-3}, etc. for $\frac{1}{a}, \frac{1}{aa}, \frac{1}{aaa}$, etc." /CAJORI (d): I, 355./

For Further Reading

BOYER (d)
CAJORI (a)
——— (d): I, 335–60
SANFORD (d): 155–58

Capsule 90 *Waldeck E. Mainville, Jr.*

RULE OF FALSE POSITION

THE rule of false position is a method of solving equations by assigning a value to the unknown; if, on checking, the given conditions are not satisfied, this value is altered by a simple proportion. For example, to solve $x + x/4 = 30$, assume any convenient value for x, say $x = 4$. Then $x + x/4 = 5$, instead of 30. Since 5 must be multiplied by 6 to give the desired 30, the correct answer must be $4 \cdot 6$ or 24.

This method was used by the early Egyptians (*c.* 1800 B.C.); many problems appearing on Egyptian papyri seem to have been solved by false position. Diophantus, in his text *Arithmetica*, uses a similar procedure to solve simultaneous equations.

The Hindu Bakhshali manuscript (*c.* A.D. 600?) contains some problems solved by false position. The earliest Arabic arithmetic of al-Khowarizmi explained the rule of false position.

The Italian mathematician Fibonacci (Leonardo of Pisa, *c.* 1200) issued a tract dealing with algebraic problems, all solved by false position. The arithmetic of Johann Widmann, published in Leipzig in 1498, is the earliest book in which the symbols + and − have been found. They occurred in connection with problems solved by false position to indicate excess and deficiency. The first edition of *Summa de arithmetica, geometrica, proportioni et proportionalita* (1494) by the Italian friar Luca Pacioli discussed and applied the rule of false position. In England Robert Recorde included the rule of false position in his arithmetic, *The Ground of Artes* (1542).

For Further Reading

MIDONICK: 91–105
SANFORD (d): 155–58
D. E. SMITH (a): II, 437–41